• Attempt all four parts. Justify your answers.

• **Note:** Rings are assumed to be commutative and with $1 \neq 0$. Modules are assumed to be unitary left modules. $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{F}_q$ denotes a finite field of $q$ elements.

**Part I.**

1. Show that if $G$ is a group of order 2023, then $G$ is an Abelian group.

2. Let $G$ be a group of order 3202 and let $C(G)$ denote the center of $G$. Show that either $G$ is cyclic or $C(G)$ is trivial. (Hint: 1601 is a prime number.)

**Part II.**

1. Given Principal Ideal Rings $A$ and $B$, show that the product-ring $A \times B$ is a Principal Ideal Ring.

2. Suppose $n$ is a positive integer and $R$ is a ring with only $n$ (distinct) maximal ideals such that $R_M (= \text{localization of } R \text{ at the maximal ideal } M)$ is a field for each maximal ideal $M$ of $R$. Show that there are fields $K_1, \ldots, K_n$ such that $R$ is isomorphic (as a ring) to the product-ring $K_1 \times \cdots \times K_n$.

**Part III.**

1. Let $R$ be a Principal Ideal Domain and let $J$ be a nonzero proper ideal of $R$. Suppose $n$ is a positive integer and $h : R^n \rightarrow \bigoplus_{1 \leq m \leq 2n} R/J^m$ is a $R$-module homomorphism. Show that $h$ is neither injective nor surjective.

2. Let $R$ be an integral domain with quotient-field $K$ and let $M$ be a $R$-submodule of $K$. For an integer $n \geq 2$, suppose the $n$-fold tensor product $M \otimes_R M \otimes_R \cdots \otimes_R M$ is a torsion-free $R$-module. Then, given a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ and $x_1, \ldots, x_n \in M$, show that

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \quad \text{(in } M \otimes_R M \otimes_R \cdots \otimes_R M).$$

**Part IV.**

1. Let $K < L$ be fields such that $[L : K] = 2$. Let $E$ be a purely transcendental field-extension of $L$ of finite transcendence degree. If the fixed-field of $G := \text{Aut}(E/K)$ is $L$, then show that $L$ is purely inseparable over $K$.

2. Let $K$ be a field and let $X$ be an indeterminate. For an integer $n$, define

$$f_n := X^3 - (4n^2 + 2n + 7) X - (4n^2 + 2n + 7) \in K[X]$$

and let $G(n, K)$ denote the Galois-group of $f_n$ over $K$. For each integer $n$, determine up to isomorphism, the groups $G(n, \mathbb{F}_2)$, $G(n, \mathbb{Q})$ and $G(n, \mathbb{F}_3)$. 
• Attempt all four parts. Justify your answers.
• For a positive integer \( n \), the group of permutations (resp. even permutations) of \( \{1, \ldots, n\} \) is denoted by \( S_n \) (resp. \( A_n \)) and \( \mathbb{Z}_n \) denotes the additive group of integers modulo \( n \).
• Rings are assumed to be commutative with \( 1 \neq 0 \) and modules are assumed to be unitary left modules.

Part I.
1. Show that a group of order 81522 is solvable but a group of order \( 8 \times 15 \times 22 \) need not be solvable.
   (Hint: 647 is a prime divisor of 81522.)
2. If a group \( G \) of order 2022 has at least 1 but at most 666 elements of order 6, then show that \( G \) is cyclic.
   (Hint: 337 is a prime divisor of 2022.)

Part II.
1. Let \( R \) be a ring and \( a, b \in R \). For a positive integer \( n \), let \( J_n := Ra^n + Rb^n \). Show that if \( J_1 \) is a principal ideal generated by a non-zerodivisor of \( R \), then \( J_n \) is a principal ideal generated by a non-zerodivisor of \( R \) for each \( n \geq 2 \). Find an example of a ring \( R \) with elements \( a, b \in R \) such that for each \( n \geq 2 \), \( J_n \) is a principal ideal generated by a non-zerodivisor of \( R \) but \( J_1 \) is not a principal ideal of \( R \).
2. Let \( R \) be a Unique Factorization Domain. Suppose \( R \) has finitely many irreducibles \( p_1, \ldots, p_n \) such that each irreducible element of \( R \) is an associate of exactly one of \( p_1, \ldots, p_n \). Show that \( R \) is a Principal Ideal Domain.

Part III.
1. Let \( R \) be a Principal Ideal Domain and suppose \( M \) is a finitely generated \( R \)-module such that \( \text{Hom}_R(\text{Hom}_R(M, R), R) \) is \( R \)-module isomorphic to \( M \). Show that \( M \) is a free \( R \)-module.
2. Let \( V \) be a vector space over \( \mathbb{Q} \). For \( v_1, v_2, v_3 \in V \), define
   \[
   f(v_1, v_2, v_3) := \sum_{\sigma \in S_3} \text{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \in V \otimes_{\mathbb{Q}} V \otimes_{\mathbb{Q}} V.
   \]
   Show that \( f(v_1, v_2, v_3) = 0 \) if and only if \( v_1, v_2, v_3 \) are \( \mathbb{Q} \)-linearly dependent.

Part IV. Let \( X \) be an indeterminate.
1. Let \( K \leq E \) be fields such that \( [E : K] = 2022 \) and \( E/K \) is Galois. Show the existence of a cubic polynomial \( f \in K[X] \) such that \( f \) is irreducible in \( K[X] \) and has 3 distinct roots in \( E \).
2. Let \( p \) be a prime number, let \( G_p \) denote the Galois group of \( X^6 - p \) over \( \mathbb{Q} \) and let
   \[
   \mathfrak{L} := \{ S_6, A_6, S_4 \times S_3, \mathbb{Z}_{12}, S_3 \times S_2, \mathbb{Z}_6, S_3 \times \mathbb{Z}_6, A_3 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_6 \}.
   \]
   Determine, with proof, the set of all \( H \in \mathfrak{L} \) such that \( H \) is isomorphic to \( G_p \) for some prime \( p \).
Algebra Preliminary Examination

January 2022

Attempt all questions, and justify each answer.

Part I

1. Let \( G \) be a group of order \( 5175 = 3^2 \cdot 5^2 \cdot 23 \). Prove that if \( H \) is a normal subgroup of order 23 in \( G \), then \( H \) is contained in the center of \( G \).

2. Let \( G \) be a group of order \( 2k \), where \( k \) is an odd positive integer. For each element \( g \in G \) let \( \sigma_g \) denote the permutation \( x \mapsto gx \) of \( G \), and let \( \Gamma = \{ \sigma_g \mid g \in G \} \).
   (a) Prove that \( \Gamma \) contains an odd permutation.
   (b) Prove that \( G \) has a subgroup of order \( k \).

Part II

1. Let \( R \) be the ring \( \mathbb{Z}[\sqrt{2}] \), consisting of all real numbers \( a + b\sqrt{2} \) with \( a, b \in \mathbb{Z} \). Prove that \( R \) is a Euclidean domain, with respect to the norm \( N(a + b\sqrt{2}) = |a^2 - 2b^2| \).

2. Let \( R \) be a commutative ring with \( 1 \neq 0 \). Prove that if every proper ideal of \( R \) is a prime ideal, then \( R \) is a field.

Part III

1. Let \( R \) be a commutative ring with \( 1 \neq 0 \). It is assumed that for each ideal \( I \) of \( R \) the quotient ring \( R/I \) is given the natural \( R \)-module structure \( r.(x + I) = (rx) + I \).
   (a) Let \( I, J \) be ideals of \( R \). Prove that \( R/I \otimes_R R/J, R/(I + J) \) are isomorphic as \( R \)-modules.
   (b) Let \( M_1, M_2 \) be distinct maximal ideals of \( R \). Prove that \( R/M_1 \otimes_R R/M_2 = 0 \).

2. Let \( R \) be the polynomial ring \( \mathbb{Z}[x] \), and let \( I = (2, x) \), the ideal of \( R \) generated by the elements 2, \( x \). Define \( R \)-module homomorphisms \( \sigma : R \to R \oplus R, \tau : R \oplus R \to I \) as follows:
   \[ \sigma(h) = (xh, -2h), \tau(f, g) = 2f + xg. \]
   (a) Prove that \( 0 \to R \xrightarrow{\sigma} R \oplus R \xrightarrow{\tau} I \to 0 \) is a short exact sequence of \( R \)-module homomorphisms.
   (b) Prove that \( I \) is not a projective \( R \)-module.

Part IV

In this part, \( x \) denotes an indeterminate.

1. Let \( f \in \mathbb{Q}[x] \) be irreducible, with splitting field \( E \) over \( \mathbb{Q} \). Assume that the degree of \( E \) over \( \mathbb{Q} \) is an odd integer, and that \( E \) contains an intermediate field \( K \) with \( [K : \mathbb{Q}] = 3 \). Prove that the irreducible factors of \( f \), considered as a polynomial over \( K \), all have the same degree.
   Hint: First show that \( K \) is a normal extension of \( \mathbb{Q} \).

2. Let \( G \) be the Galois group of the polynomial \( f = x^4 + 2x^2 + 4 \in \mathbb{Q}[x] \). Determine the order of \( G \), and describe how each element of \( G \) permutes the roots of \( f \).
Algebra Preliminary Examination
August 2021

Attempt all questions, and justify each answer.

Part I

1. Let $G$ be a group. Recall that the \textit{commutator subgroup} $[G, G]$ of $G$ is the subgroup generated by all commutators $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ $(g_1, g_2 \in G)$. Also recall that a subgroup $H$ of $G$ is characteristic in $G$, written $H \text{ char } G$, if each automorphism of $G$ maps $H$ onto itself.

(a) Define subgroups $G^{(n)}$ $(n \in \mathbb{Z}, n \geq 0)$ inductively as follows:

$$G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Prove that $G^{(n)} \text{ char } G$ for all $n \geq 0$.

(b) Suppose that $G$ is a non-trivial finite group, such that $G^{(n)} = 1$ for some $n > 0$. Prove that $G$ has a non-trivial characteristic subgroup of prime power order. (\textit{Hint}: consider the subgroup $G^{(n-1)}$, where $n$ is the smallest integer for which $G^{(n)} = 1$.)

2. The \textit{holomorph} of a group $G$, denoted $\text{Hol}(G)$, is defined to be the semidirect product $G \rtimes \text{Aut}(G)$, where $\phi : \text{Aut}(G) \to \text{Aut}(G)$ is the identity map. Thus we may identify $\text{Aut}(G)$ with the subgroup $K = \{ (1, \sigma) : \sigma \in \text{Aut}(G) \}$ of the semidirect product $\text{Hol}(G)$. As usual we identify $G$ with the (normal) subgroup $\{ (g, 1) : g \in G \}$ of $\text{Hol}(G)$.

Let $G = \{1, z_1, z_2, z_3\}$ be the non-cyclic group of order 4 (\textit{i.e.} $G$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). Prove that $\text{Hol}(G)$ is isomorphic to the symmetric group $S_4$. (\textit{Hint}: Consider the action by left multiplication of $\text{Hol}(G)$ on the four left cosets $K, z_1K, z_2K, z_3K$ of $K$.)

Part II

1. Let $R$ be an integral domain with the property that every ideal generated by two elements of $R$ is principal.

(a) Prove that every finitely generated ideal of $R$ is principal.

(b) Suppose that $R$ also satisfies the ascending chain condition on principal ideals, \textit{i.e.} given any chain of principal ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$, there exists a positive integer $k$ such that $I_k = I_{k+n}$ for all positive integers $n$. Prove that $R$ is a principal ideal domain.

2. Recall that an element $e$ of a ring $R$ is \textit{idempotent} if $e^2 = e$. In this question all rings are assumed to be commutative and with $1 \neq 0$.

(a) Let $R$ be a ring containing an idempotent $e$ distinct from 0, 1. Prove that $R$ is isomorphic to a direct product of two rings. (\textit{Hint}: if $e$ is idempotent, then so is $1 - e$.)

(b) Suppose that $R$ is a finite ring and that every element of $R$ is idempotent. Prove that $R$ is isomorphic to the direct product of finitely many copies of the field with two elements.
Part III  

In this part, all $R$–modules $M$ are assumed to be unital, i.e. $1:m$ for all $m \in M$.

1. Recall that given left $R$–modules $D, M, N$, an $R$–module homomorphism $\phi : M \to N$ induces a homomorphism of Abelian groups $\phi' : \text{Hom}_R(D, M) \to \text{Hom}_R(D, N)$ given by $\phi'(\alpha) = \phi \circ \alpha$.

Let $R$ be a ring with $1 \neq 0$ and let $D, L, M, N$ be left $R$–modules. Prove that if the sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

of module homomorphisms is exact, then the sequence of induced homomorphisms of Abelian groups

$$0 \to \text{Hom}_R(D, L) \xrightarrow{\phi'} \text{Hom}_R(D, M) \xrightarrow{\psi'} \text{Hom}_R(D, N)$$

is also exact.

2. Let $I = (2, x)$ be the ideal generated by 2 and $x$ in the ring $R = \mathbb{Z}[x]$, $x$ being an indeterminate. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ inherits from $R$ a natural $R$–module structure, with annihilator $I$.

(a) Show that there is an $R$–module homomorphism from $I \otimes_R I$ to $\mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2} q'(0)$, where $q'$ denotes the usual polynomial derivative of $q$.

(b) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Part IV  

In this part, $x$ denotes an indeterminate.

1. This question concerns the polynomial $f(x) := x^{p^n} - x + 1 \in \mathbb{F}_p[x]$ ($n \geq 1$). We take some fixed algebraic closure $\mathbb{A}$ of $\mathbb{F}_p$, and denote by $\mathbb{F}_{p^n}$ the unique field of order $p^n$ contained in $\mathbb{A}$. You may assume that each extension of finite degree of $\mathbb{F}_p$ is Galois over $\mathbb{F}_p$, with cyclic Galois group generated by the Frobenius automorphism $\phi : a \mapsto a^p$.

(a) Let $E$ be the splitting field over $\mathbb{F}_p$ of $f(x) = x^{p^n} - x + 1$ in $\mathbb{A}$. Show that $E$ contains $\mathbb{F}_{p^n}$ as a subfield. *(Hint: If $\alpha$ is a root of $f(x)$, then so is $\alpha + a$ for each $a \in \mathbb{F}_{p^n}$.)

(b) Determine the order of the Frobenius automorphism $\phi : E \to E$, $\phi : \beta \mapsto \beta^p$. *(Hint: First compute $\phi^n(\alpha)$, where $\alpha$ is a root of $f(x)$.)

(c) Show that if $f(x)$ is irreducible over $\mathbb{F}_p$, then $pn = p^n$.

[Observation (you may omit the easy proof): from $pn = p^n$ it follows that $n = 1$ or $n = p = 2$.]

2. Determine the Galois group over $\mathbb{Q}$ of $x^4 + 9$, describing how each automorphism permutes the roots of this polynomial.
Part I

1. Let \( p \) be a prime, and let \( S_p \) denote the symmetric group of degree \( p \). Prove that if \( P \) is a subgroup of \( S_p \) of order \( p \), then the normalizer of \( P \) in \( S_p \) has order \( p(p - 1) \).

2. Classify, up to isomorphism, the groups of order 63.

Part II

1. A local ring is a commutative ring with \( 1 \neq 0 \) that has a unique maximal ideal. Prove that if \( R \) is a local ring with maximal ideal \( M \), then every element of \( R \setminus M \) is a unit. Also prove that if \( R \) is a commutative ring with \( 1 \neq 0 \), in which the set of nonunits forms an ideal \( M \), then \( R \) is a local ring with maximal ideal \( M \).

2. Let \( p \in \mathbb{Z}_+ \) be prime, and let \( \mathbb{Z}[i] \) denote the usual ring of Gaussian integers \( \{ a + bi \mid a, b \in \mathbb{Z} \} \). For which \( p \) is the quotient ring \( \mathbb{Z}[i]/(p) \) (i) a field, (ii) a product of fields? Justify your answer. (You may use the following facts: (i) \( \mathbb{Z}[i] \) is a Euclidean Domain with respect to the field norm, hence is also a Unique Factorization Domain, and (ii) a prime \( p \in \mathbb{Z}_+ \) with \( p \equiv 1 \mod 4 \) can be written as the sum of two integer squares.)

Hint: Use the Chinese Remainder Theorem where appropriate. Also note that a product of fields cannot contain a nonzero nilpotent element.

Part III

1. Let \( V \) be a finite dimensional vector space over a field \( F \), and let \( v_1, v_2 \) be nonzero elements of \( V \). Prove that \( v_1 \otimes v_2 = v_2 \otimes v_1 \) in \( V \otimes_F V \) if and only if \( v_1 = \lambda v_2 \) for some \( \lambda \in F \).

2. Let \( R \) be a ring with \( 1 \neq 0 \), let \( P, M, N \) be \( R \)-modules, and let there be an exact sequence of \( R \)-module homomorphisms \( M \to N \to 0 \).

(a) Prove that if \( P \) is a direct summand of a free \( R \)-module, then the induced sequence of Abelian group homomorphisms

\[
\text{Hom}_R(P, M) \xrightarrow{\phi^*} \text{Hom}_R(P, N) \to 0
\]

is exact. (Here \( \phi^* \) is the homomorphism \( \psi \mapsto \phi \circ \psi \).)

(b) Show by means of an example that in general the induced sequence \( \text{Hom}_R(P, M) \xrightarrow{\phi^*} \text{Hom}_R(P, N) \to 0 \) need not be exact.

Note: For this question do not assume any result concerning projective modules.
Part IV: In this part, \( x \) denotes an indeterminate.

1. This question concerns the splitting field over \( \mathbb{Q} \) of the polynomial \( x^4 - 2x^2 - 2 \in \mathbb{Q}[x] \).
   
   (a) Prove that \( x^4 - 2x^2 - 2 \) is irreducible over \( \mathbb{Q} \), and that its roots in \( \mathbb{C} \) are \( \pm \alpha, \pm \beta \), where \( \alpha = \sqrt{1 + \sqrt{3}}, \beta = \sqrt{1 - \sqrt{3}} \).

   (b) Prove that \( \mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta) \), and that \( [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2 \).

   (c) Prove that the splitting field of \( x^4 - 2x^2 - 2 \) has degree 8 over \( \mathbb{Q} \), and that the Galois group of this polynomial over \( \mathbb{Q} \) is dihedral of order 8.

   *Hint for (c):* The Galois group acts faithfully on the set of roots of the polynomial.

2. Let \( \mathbb{F}_p \) denote the field of order \( p \), let \( f \in \mathbb{F}_p[x] \) be irreducible over \( \mathbb{F}_p \), and let \( K \) be a splitting field for \( f \) over \( \mathbb{F}_p \).

   Let \( L \) be an intermediate field, i.e. \( \mathbb{F}_p \subseteq L \subseteq K \). Prove that the irreducible factors of the polynomial \( f \) in \( L[x] \) all have the same degree.

   *Hint:* Here is one approach. Let \( g \in L[x] \) be a factor of \( f \) that is irreducible in \( L[x] \), and let \( \alpha \) be a root of \( g \) in \( K \). Consider the relationship between \( [L(\alpha) : L] \) and \( [K : L] \).
Algebra Preliminary Examination
August 2020

Attempt all questions, and justify each answer.

Part I

1. Let $P$ be a Sylow $p$–subgroup of a finite group $G$. If $p$ is the smallest prime dividing $|G|$ and $P$ is cyclic, prove that $N_G(P) = C_G(P)$. (Recall that $N_G(P), C_G(P)$ denote the normalizer and centralizer of $P$ in $G$, respectively.)

(Hint: Consider the order of the automorphism group of $P$ and the action of $N_G(P)$ on $P$ by conjugation.)

2. (a) Prove that a group of order 105 contains a cyclic normal subgroup of order 35.

(b) Prove that, up to isomorphism, there is just one non-Abelian group of order 105.

In parts II, III and IV, $X$ denotes an indeterminate.

Part II

1. Let $R$ be a commutative ring with $1 \neq 0$. Recall that $R$ is Artinian if it satisfies the descending chain condition on ideals, i.e. if $I_1 \supseteq I_2 \supseteq \ldots$ is a descending chain of ideals of $R$, then there exists $k \in \mathbb{Z}_+$ such that $I_m = I_k$ for all $m > k$.

Let $S$ be an arbitrary commutative ring with $1 \neq 0$, and let $J$ denote the Jacobson radical of $S[X]$. Prove that $S[X]/J$ is not Artinian.

2. Let $R$ be the subring of $\mathbb{Q}[X]$ consisting of all polynomials whose constant term is an integer.

(a) Prove that $R$ is an integral domain in which every irreducible element is prime.

(b) Prove that $R$ is not a Unique Factorization Domain.

(Hint: Consider factorizations of the element $X$.)

Part III

1. Let $k$ be a field, and let $R = M_2(k)$ be the ring of $2 \times 2$ matrices over $k$. Let $P$ be the set of $2 \times 1$ matrices over $k$: then $P$ is an Abelian group under matrix addition, and left matrix multiplication of elements of $P$ by elements of $R$ accords $P$ the structure of a left $R$–module.

Prove that the $R$–module $P$ is projective, but not free.

2. Let $R = \mathbb{Z}[X]$, let $I \subset R$ be the ideal generated by $2, X$, and let $M = I \otimes_R I$.

Prove that the element $2 \otimes 2 + X \otimes X \in M$ cannot be written as a simple tensor $a \otimes b$ ($a, b \in I$).

(Hint: Use a suitable $R$–module homomorphism defined on $M$.)
Part IV

1. Prove that $\mathbb{Q}(\sqrt{5} + 2\sqrt{5})$ is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

2. Let $F$ be a field (possibly infinite) of finite characteristic $p$, and let $a \in F$ be an element not of form $b^p - b$ for any $b \in F$. Let $f = X^p - X - a \in F[X]$.

   (a) Prove that the polynomial $f$ is separable and irreducible over $F$.

   (b) Prove that if $\alpha$ is a root of $f$ in an extension field of $F$, then $F(\alpha)$ is a splitting field for $f$ over $F$.

   (Hint: Consider the effect of substituting $X + 1$ for $X$ in the polynomial $f$.)
ALGEBRA PRELIMINARY EXAM

JANUARY 2020

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ be a finite group and $\phi : G \to H$ a surjective homomorphism. Prove that if $y \in H$ is such that $|y| = p^r$, for some prime $p$ and $r \in \mathbb{Z}_{>0}$, then there is $x \in G$ such that $\phi(x) = y$ and $|x| = p^s$, for some $s \in \mathbb{Z}_{>0}$.
   [Hint: Let $g \in G$ such that $\phi(g) = y$, and write $|g| = n \cdot p^k$, where $p \nmid n$.]

2. Let $G$ be a group of order 60 and assume that 4 divides $|Z(G)|$ [where $Z(G)$ denotes the center of $G$]. Prove that $G$ must be Abelian.

Part II

1. Let $I$ be the ideal of $\mathbb{Z}[x]$ generated by 7 and $x^2 + 1$. Prove that $I$ is a maximal ideal.

2. Let $R$ be an integral domain such that for any descending chain of ideals
   \[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \]
   there is a positive integer $N$ such that $I_i = I_N$ for all $i \geq N$. Prove that $R$ is a field.

Part III

1. Let $R$ be a subring of $S$. Prove that $S \otimes_R S \neq 0$.

2. Let $R$ be a ring containing $\mathbb{Z}$ such that $R$ is a free $\mathbb{Z}$-module of finite rank $n > 0$ and every non-zero ideal of $R$ has a non-zero element of $\mathbb{Z}$. Prove that for every non-zero ideal $I$ we have that $R/I$ is finite.

Part IV

1. Given a prime $p$ and a positive integer $n$, show that there is an Abelian extension [i.e., Galois with Abelian Galois group] $K$ of $\mathbb{Q}$ with $[K : \mathbb{Q}] = p^n$.

2. Let $F$ be a field of characteristic $p$ with exactly $p^r$ elements. If $K$ is a finite extension of $F$ with $K = F[\alpha]$, for some $\alpha \in K$, and $f$ is the minimal polynomial of $\alpha$ over $F$, then show that if $\beta$ is another root of $f$, then $\beta \in K$ and $\beta = \alpha^p^k$ for some $k \in \mathbb{Z}$. 
ALGEBRA PRELIMINARY EXAM

AUGUST 2019

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have 1 ≠ 0 [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G_1, G_2$ be groups, $N \trianglelefteq G_1$, and $\phi : G_1 \to G_2$ be an onto homomorphism such that $N \cap \ker(\phi) = \{1\}$. Prove that for $x \in N$ we have that $C_{G_2}(\phi(x)) = \phi(C_{G_1}(x))$. [Remember: $C_G(x) \overset{\text{def}}{=} \{g \in G : gx = xg\}$ is the centralizer of $x$ in $G$.]

2. Let $G$ be a group of order $992 = 2^5 \cdot 31$. Prove that either $G$ has a normal subgroup of order $32 = 2^5$ or it has a normal subgroup of order $62$.

Part II

1. Let $R$ be a UFD with exactly two non-associate prime elements $p$ and $q$ [i.e., $p$ and $q$ are non-associate primes and every prime is an associate of either $p$ or $q$]. Prove that $R$ is a PID.

2. Let $R$ be a PID and $P$ a prime ideal of $R[x]$ such that $P \cap R \neq \{0\}$. Prove that there is $p \in R$ prime [in $R$] such that either $P = (p)$ or $P = (p, f)$ for some $f$ prime in $R[x]$.

Part III

1. Let $R$ be a commutative ring and $M$ an $R$-module. Prove that $R \otimes_R \text{Hom}_R(R \oplus R, M)$ is projective if and only if $M$ is projective.

2. Let $R$ be a commutative ring, $M$ and $N$ be $R$-modules and $M'$ and $N'$ be submodules of $M$ and $N$ respectively. Define $L$ as the submodule of $M \otimes_R N$ generated by the set

\[ \{x \otimes y \in M \otimes_R N : x \in M' \text{ or } y \in N'\}. \]

Show that $M/M' \otimes_R N/N' \cong (M \otimes_R N)/L$.

Part IV

1. Let $F = \mathbb{Q}(\sqrt[3]{2} \cdot \zeta)$, where $\zeta = -1/2 + \sqrt{3}i/2$ [a primitive third root of unity]. Prove that $-1$ is not a sum of squares in $F$, i.e., there is no positive integer $n$ and $\alpha_1, \ldots, \alpha_n \in F$ such that $-1 = \alpha_1^2 + \cdots + \alpha_n^2$.

2. Let $F$ be a field of characteristic 0 and $K/F$ be a field extension of degree $n$ such that there is a root of unity $\zeta$ in the algebraic closure of $K$ such that $K \subseteq F[\zeta]$. Prove that if $d \mid n$, there is $\alpha \in K$ such that the minimal polynomial of $\alpha$ over $F$ has degree $d$. 
ALGEBRA PRELIMINARY EXAM

AUGUST 2018

Instructions: Attempt all problems in all four parts. Justify your answers.

General assumptions: All rings have 1 ≠ 0, their subrings contain 1, and all modules are unitary.

Part I

1. Let $G$ be a (possibly infinite) group, and suppose that $G$ contains a subgroup $H \neq G$ whose index $[G : H]$ is finite. Prove that $G$ contains a normal subgroup $N \neq G$ of finite index.
2. Prove that every group of order 70 contains a cyclic, normal subgroup of order 35.

Part II

1. Let $R$ be a commutative ring in which every element is either a unit or nilpotent. Prove that $R$ has exactly one prime ideal.
2. If $R$ is an integral domain, prove that there are infinitely many ideals in $R[x]$ that are both prime and principal.

Part III

1. Let $R$ be a ring, possibly non-commutative, and suppose that
   
   $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$

   is a short exact sequence of left $R$-modules, with $M'$ and $M''$ finitely generated. Prove that $M$ is finitely generated.
2. Let $M$ be a finitely-generated $\mathbb{Z}$-module, and let $T \subseteq M$ be its torsion submodule. Show that the torsion submodule of $M \otimes_{\mathbb{Z}} M$ has at least $|T|$ elements.

Part IV

1. Let $p$ be a prime and suppose that $f \in \mathbb{F}_p[x]$ is an irreducible polynomial. Let $K$ be a degree 2 extension of $\mathbb{F}_p$ and suppose that there exist non-constant polynomials $g, h \in K[x]$ such that $f = gh$. If $g$ is an irreducible polynomial of degree 5, what is the degree of $f$?
2. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial, and $K/\mathbb{Q}$ is an extension such that $f$ has exactly one root in $K$. Let $G$ be the Galois group of $f$, and show that $|G|$ is divisible by 12.
ALGEBRA PRELIMINARY EXAM

AUGUST 2017

Instructions: Attempt all problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Suppose that $H$ is a subgroup of a finite group $G$ of index $p$, where $p$ is the smallest prime dividing the order of $G$. Prove that $H$ is normal in $G$.

2. Show that every group of order 222 is solvable.

Fun fact: The University of Tennessee was established 222 years ago.

Part II

1. Let $I$ and $J$ be ideals of a ring $R$ and assume that $P$ is a prime ideal of $R$ that contains $I \cap J$. Prove that either $I$ or $J$ is contained in $P$.

2. Let $R$ be an integral domain and suppose that every prime ideal in $R$ is principal. Prove that $R$ is a PID.

Part III

1. Let $V$ be a Noetherian right $R$-module, and $\theta : V \to V$ a homomorphism.
   (a) Show that $\ker(\theta^{n+1}) = \ker(\theta^n)$ for some $n \geq 1$.
   (b) If $\theta$ is onto, show that it is one-to-one.

2. An $R$-projection is defined to be an $R$-module homomorphism $\varphi : R^n \to R^n$ such that $\varphi^2 = \varphi$. Prove that a finitely generated $R$-module $M$ is projective if and only if it is isomorphic to the image of some $R$-projection.

Part IV

1. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F(\alpha)$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   (a) If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$.
   (b) If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$.
   (c) If every root of unity of $E$ lies in $F$, show that $[E : F] = n$.

2. Let $F$ be a field of characteristic 0 and let $E$ be a finite Galois extension of $F$.
   (a) If $0 \neq \alpha \in E$ with $E = F(\alpha)$, show that $F(\alpha^2) \neq E$ if and only if there exists $\sigma \in \text{Gal}(E/F)$ with $\sigma(\alpha) = -\alpha$.
   (b) Prove that there exists an element $\alpha \in E$ with $E = F(\alpha^2)$.
• Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 255 is not a simple group.

2. A group $G$ has a cyclic normal subgroup of order 2016. If $G$ also has a subgroup of order 2017, then show that $G$ has a cyclic subgroup of order $(2016) \times (2017)$.

Part II.

**Note**: Rings are assumed to be commutative and with $1 \neq 0$.

1. Let $A$ and $B$ be rings. Show that each ideal of $A \times B$ is of the form $I \times J$, where $I$ is an ideal of $A$ and $J$ is an ideal of $B$.

2. Let $R$ be a ring, let $X$ be an indeterminate and let $S := \{X^n \mid 0 \leq n \in \mathbb{Z}\}$. If $S^{-1}R[[X]]$ is a field, then show that $R$ is a field.

Part III.

**Note**: Rings are assumed to be commutative with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $A$ be a ring and let $M$, $N$ be finitely generated projective (left) $A$-modules. Show that $\text{Hom}_A(M, N)$ is a finitely generated projective $A$-module.

2. Let $R$ be a PID and let $I$, $J$ be ideals of $R$. If $I \neq R \neq J$, then show that $(R/I) \oplus (R/J)$ and $(R/I) \otimes_R (R/J)$ are not isomorphic as (left) $R$-modules.

Part IV.

**Note**: In what follows, $X$ is an indeterminate.

1. Let $K$ be an extension-field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}_{30}$. Suppose each of $f, g \in \mathbb{Q}[X]$ is an irreducible polynomial of degree 6 and $f$ has a root $a \in K$. If $g$ has a root in $K$, then show that $g$ has all its roots in $\mathbb{Q}[a]$.

2. Let $F \subset K$ be finite fields of characteristic 5 and suppose $g \in F[x]$ is irreducible in $F[x]$. If $g$ has degree 11, then show that either $g$ is irreducible in $K[x]$ or all its roots are in $K$. 
• Attempt all four parts. Justify your answers.

Part I.

1. Let $p$ be a prime number and $G$ be a non-Abelian group of order $p^3$. Show that $G$ has at least 3 (distinct) subgroups of index $p$.

2. Let $G$ be a group of order $p^3q$, where $p$, $q$ are distinct prime numbers. If no Sylow $p$-subgroup of $G$ is normal and also no Sylow $q$-subgroup of $G$ is normal, then show that $G$ has order 24.

Part II.

Note: *Rings are tacitly assumed to be commutative and with $1 \neq 0$.*

1. Let $R$ be a ring, $X$ an indeterminate and $h : R[X] \rightarrow R[[X]]$ a ring-homomorphism such that $h(a) = a$ for all $a \in R$. Show that $h$ is not surjective.

2. Let $R$ be an integral domain with at least 3 (distinct) maximal ideals. Given maximal ideals $M$ and $N$ of $R$, show that $R_M \cap R_N \neq R$. (Here localization of $R$ at a prime ideal is naturally identified as a ring in between $R$ and the quotient-field of $R$.)

Part III.

Note: *Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.*

1. Let $R$ be a ring and let $a \in R$ be a nonzero element of $R$ such that $a^3 = a$. Show that the ideal $Ra$ is a projective $R$-module.

2. Let $R$ be a PID and let $M$ be a finitely generated $R$-module. For a maximal ideal $Q$ of $R$, let $\delta(Q, M)$ denote the dimension of $M \otimes_R R/Q$ as a vector-space over the field $R/Q$. Let $\delta(M)$ denote the sup\{\$\delta(Q, M)\$\}, where the supremum is taken over all maximal ideals $Q$ of $R$. Show that as an $R$-module, $M$ has a generating set of cardinality $\delta(M)$ and any generating set of $M$ has cardinality at least $\delta(M)$.

Part IV.

Note: *In what follows, $X$ is an indeterminate.*

1. Let $f(X)$ be a monic polynomial with rational coefficients. Assume $f(X)$ is irreducible in $\mathbb{Q}[X]$ and the Galois-group of $f(X)$ over $\mathbb{Q}$ is a group of order 99. What is the degree of $f(X)$?

2. Compute the Galois group of $X^6 - 9$ over $\mathbb{Q}$.
ALGEBRA PRELIMINARY EXAM

JANUARY 2016

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ be a finite group and $H$ be a subgroup of $G$. Prove that $n_p(H) \leq n_p(G)$, where $n_p(X)$ denotes the number of Sylow $p$-subgroups of $X$.

2. Let $G$ be a group of order $p^n$ for some prime $p$ and positive integer $n$. Prove that if $1 \neq H \leq G$, then $Z(G) \cap H \neq 1$. [Here $Z(G)$ denotes the center of $G$.]

Part II

1. Let $R$ be a Boolean ring, i.e., a ring [with 1] for which $a^2 = a$ for all $a \in R$. [You can use without proof the well known fact that if $R$ is Boolean, then it is commutative of characteristic 2.]
   (a) Prove that if $R$ is finite, then its order is a power of 2.
   (b) Prove that every prime ideal of $R$ is maximal.

2. Show that $R \overset{\text{def}}{=} \mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1x_2, x_3x_4, x_5x_6, \ldots)$ has infinitely many distinct minimal prime ideals. [$P$ is a minimal prime ideal if it is prime and whenever $Q \subseteq P$, with $Q$ also prime, we have $Q = P$.]

Part III

1. Let $F$ be a field and $M$ be a torsion $F[x]$-modulo. Prove that if there is $m_0 \in M$, with $m_0 \neq 0$, and an irreducible $f \in F[x]$ such that $f \cdot m_0 = 0$, then $\text{Ann}(M) \subseteq (f)$.

2. Let $R$ be an integral domain and $I$ a principal ideal of $R$. Prove that $I \otimes_R I$ has no non-zero torsion element [i.e., if $m \in I \otimes_R I$, with $m \neq 0$, and $r \in R$ with $rm = 0$, then $r = 0$].

Part IV

1. Let $K/F$ be an algebraic field extension and $\text{Emb}(K/F)$ denote the set of field homomorphisms $\sigma : K \to \overline{K}$ such that $\sigma(a) = a$ for all $a \in F$. [Here $\overline{K}$ is a fixed algebraic closure of $K$.]
   (a) Prove that if $\alpha$ is a root of a [not necessarily irreducible] non-zero polynomial $f \in F[x]$ with $\deg(f) = n$, then $\text{Emb}(F[\alpha]/F)$ has at most $n$ elements.
   (b) Give an example of an algebraic extension $K/F$ of degree greater than one for which $\text{Emb}(K/F)$ has a single element.

2. Let $F = \mathbb{Q}[\sqrt{2}]$ and $K = \mathbb{Q}[\sqrt[3]{2}, i]$.
   (a) Prove that $K/F$ is Galois with $[K : F] = 8$.
   (b) Prove that $\text{Gal}(K/F)$ has a non-normal subgroup. [This implies that it is the dihedral group of order 8, as it is the only group of order 8 with this property.]
ALGEBRA PRELIMINARY EXAM

AUGUST 2015

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have 1 ≠ 0 [and their subrings contain 1] and all modules are unitary.

Part I
1. Let G be a non-Abelian group of order $p^3$, $[G, G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle$ be its commutator subgroup and $Z(G)$ be its center. Show that $|Z(G)| = p$ and that $Z(G) = [G, G]$.

2. Let $G_1$ and $G_2$ be groups of order 81 acting faithfully [i.e., only 1 acts as the identity function] on sets $X_1$ and $X_2$, respectively, with 9 elements each. Show that there is an isomorphism $\psi : G_1 \rightarrow G_2$.

Part II
1. Let $D$ be a finite division ring. Prove that $D$ has a prime power number of elements. [Hint: Consider the center $Z(D) = \{a \in D : ax = xa \text{ for all } x \in D\}$]

2. Let $p \in \mathbb{Z}$ prime and

$$f = a_{2n+1}x^{2n+1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x].$$

Prove that if $p^3 \nmid a_0$, $p^2 \mid a_0, a_1, \ldots, a_n$, $p \mid a_{n+1}, a_{n+2}, \ldots, a_{2n}$ and $p \nmid a_{2n+1}$, then $f$ is irreducible in $\mathbb{Q}[x]$.

Part III
1. Let $R$ be a commutative ring. An $R$-module is Artinian if it satisfies the descending chain condition for submodules. [i.e., if $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ is a chain of submodules, then there is a $i_0$ such that for all $i \geq i_0$, we have $S_i = S_{i_0}$] Show that if $L$ and $N$ are Artinian $R$-modules and we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0,$$

then $M$ is also Artinian.

2. Let $R$ be a commutative ring such that every $R$-module is free. Prove that $R$ is a field.

Part IV
1. Let $\mathbb{F}_p$ be the field with $p$ elements, and $t$ be an indeterminate. Let $f(t), g(t) \in \mathbb{F}_p[t] \setminus \{0\}$, with $\max\{\deg f, \deg g\} < p$ and $f(t)/g(t) \notin \mathbb{F}_p$. Show that the extension $\mathbb{F}_p(t)/\mathbb{F}_p(f(t)/g(t))$ is separable.

2. Suppose that $f = \prod_{i=1}^{N}(x - \alpha_i) \in \mathbb{Q}[x]$ [with $\alpha_i \in \mathbb{C}$] is irreducible in $\mathbb{Q}[x]$ and let $f_n \overset{df}{=} \prod_{i=1}^{N}(x - \alpha_i^n)$. Prove that for each $n$, there is $g_n \in \mathbb{Q}[x]$ irreducible and a positive integer $k_n$ such that $f_n = g_n^{k_n}$.
ALGEBRA PRELIMINARY EXAMINATION
Fall 2014

Attempt all four parts. Justify your answers.

Part I.

1. Show that $S_4$ (the group of permutations of $\{1, 2, 3, 4\}$) does not have a subgroup isomorphic to $Q_8$ (the quaternion-group of order 8).

2. Let $G$ be a group of order 2014. Show that $G$ is cyclic if and only if $G$ has a normal subgroup of order 2.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be an integral domain with only finitely many units. Show that the intersection of all maximal ideals of $R$ is 0.

2. Let $R$ be a ring such that each non-unit of $R$ is nilpotent. Let $X$ be an indeterminate and let $f \in R[[X]]$. Show that $f^n = f$ for some integer $n \geq 2$ if and only if either $f = 0$ or $f^{n-1} = 1$.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $L$ be a module over a ring $R$ and let $M, N$ be $R$-submodules of $L$. Show that if $(M + N)/(M \cap N)$ is a projective $R$-module then $M/(M \cap N)$ is also a projective $R$-module.

2. Let $R$ be a PID with infinitely many prime ideals and let $M$ be a finitely generated $R$-module. Show that $M$ is a torsion $R$-module if and only if $M \otimes_R R/P = 0$ for all but finitely many prime ideals $P$ of $R$.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $f(X) := X^5 + 3X^3 + X^2 + 3 \in \mathbb{Q}[X]$. Let $K$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.

2. Let $f(X) := X^3 + X + 1 \in \mathbb{Q}[X]$. Let $F$ be a finite Galois extension of $\mathbb{Q}$ such that the Galois group of $F$ over $\mathbb{Q}$ is an Abelian group. Show that $f$ is irreducible in $F[X]$. 
Algebra Preliminary Exam    January 2014

Attempt all problems and justify all your answers. All rings have a 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I. Groups

1. Show that every group of order 1,225 is abelian.
2. Let n ≥ 2. Show that there is a nontrivial homomorphism 
   \[ f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \]  
   (i.e., \( \ker f \neq \mathbb{Z}/n\mathbb{Z} \)) if and only if n is even.

Part II. Rings

1. Let R be a commutative ring. Show that \( J(R[X]) = \text{nil}(R[X]) \).
   
   (\( J(A) \) and \( \text{nil}(A) \) are the Jacobson and nil radicals of A.)
2. Let R be a PID.
   (a) Show that R satisfies ACC on ideals.
   (b) Show that every nonzero prime ideal of R is maximal.

Part III. Modules

1. Let R be a ring and M a nonzero R-module. Show that 
   \( M = A \oplus B \) for proper submodules A and B of M if and only if 
   there is a nonzero, nonidentity homomorphism \( f : M \rightarrow M \) 
   with \( f^2 = f \).
2. Let R be a commutative ring, I a proper ideal of R, and M an R-module. Show that \( (R/I) \oplus_{R} M \) and \( M/IM \) are isomorphic as R-modules.

Part IV. Fields

1. Let K a subfield of a field F. Show that there is a subring of F containing K that is a PID, but not a field, 
   if and only if the extension F/K is not algebraic.
2. Determine the Galois group of \( f(X) = X^{10} + X^{8} + X^{6} + X^{2} \) 
   over \( \mathbb{Z}/2\mathbb{Z} \).
Algebra Preliminary Exam

August 2013

Attempt all problems and justify all your answers. All rings have an identity 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I.

1. (a) Let p and q be (not necessarily distinct) prime numbers. Show that a group G with \( |G| = pq \) is either abelian or \( Z(G) = \{e\} \).
   
   (b) Give an example of a nonabelian group G whose order is the product of three (not necessarily distinct) primes and \( Z(G) \neq \{e\} \).

2. (a) Let G be a group with \( |G| = 100 \). Show that G is abelian if and only if its Sylow 2-subgroup is normal.
   
   (b) Give an example of a nonabelian group of order 100.

Part II.

1. Let R and S be a commutative rings with 1 ≠ 0. Show that every ideal of \( R \times S \) has the form \( I \times J \) for I an ideal of R and J an ideal of S.

2. Let R be a commutative ring with 1 ≠ 0. Show that \( f(X) = a_0 + a_1X + \cdots + a_nX^n \) is a unit in \( R[X] \) if and only if \( a_0 \) is a unit in R and \( a_1, \ldots, a_n \) are nilpotent.
Part III

1. Let $P$ and $Q$ be finitely generated projective $R$-modules over a commutative ring $R$ with $1 \neq 0$. Show that $\text{Hom}_R(P, Q)$ is a finitely generated projective $R$-module.

2. Let $R$ be a commutative ring with $1 \neq 0$, $S$ a nonempty multiplicatively closed subset of $R$, and $M$ an $R$-module. Show that $(S^{-1}R) \otimes_R M$ and $S^{-1}M$ are isomorphic as $S^{-1}R$-modules.

Part IV.

1. Let $p$ and $q$ be distinct prime numbers, $F$ a subfield of a field $K$, and $f(X)$, $g(X) \in F[X]$ be irreducible with $\deg(f(X)) = p$ and $\deg(g(X)) = q$. Let $a, b \in K$ be roots of $f(x)$ and $g(X)$, respectively. Show that $[F(a,b):F] = pq$.

2. (a) Let $F$ be a splitting field for $f(X) \in \mathbb{Q}[X]$ over $\mathbb{Q}$ with abelian Galois group $G$. Show that every subfield $L$ of $F$ is a splitting field over $\mathbb{Q}$ for some polynomial $g(X) \in \mathbb{Q}[X]$.

(b) Give an example to show that if $G$ is not abelian in part (a), then some $L$ need not be a splitting field.
ALGEBRA PRELIMINARY EXAM

JANUARY 2013

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have 1 ≠ 0 [and their subrings contain 1] and all modules are unitary.

Part I

1. Let p and q be prime numbers such that q < p and q does not divide p^2 - 1. Prove that every group of order p^2q is Abelian.

2. Let G be a finite simple group. Show that if p is the largest prime dividing |G|, then there is no subgroup H ≤ G such that 1 < |G : H| < p.

Part II

1. Let R be a ring not necessarily having 1 [or commutative], with at least two elements and such that for all non-zero a ∈ R there is a unique b ∈ R such that aba = a.
   (a) Show that R has no [non-zero] zero divisors.
   (b) Show that for a and b as above, we also have bab = b.
   (c) Show that R has 1.

2. Let R be a commutative ring and a ∈ R such that a^n ≠ 0 for all positive integers n. Let I be an ideal maximal with respect to the property that a^n ∉ I for any positive integer n. Show that I is prime.

Part III

1. Let V = ℝ^2 and {e_1, e_2} be a basis of V. Show that e_1 ⊗ e_2 + e_2 ⊗ e_1 ∈ V ⊗_ℝ V cannot be written as a single tensor.

2. Let R be a PID.
   (a) Prove that a finitely generated R-module M is free if and only if it is torsion free.
   (b) Prove that if a finitely generated R-module M is projective, then it is free.

Part IV

1. Let ℱ_p be the field with p elements, ℱ_p be a fixed algebraic closure of ℱ_p and let
   \[ L = \{ α ∈ ℱ_p : p ∤ [𝔽_p[α] : ℱ_p] \}. \]
   Show that L is a field.

2. Let p be a prime, F be a field of characteristic different from p and f = x^p - a ∈ F[x] [not necessarily irreducible]. Let K be the splitting field of x^p - 1 over F and assume that all roots of f lie in K.
   (a) Show that if f(α) = 0 with α ∉ F, then F[α] = K.
   (b) Prove that f has a root in F.
ALGEBRA PRELIMINARY EXAM

AUGUST 2012

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: All rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ and $H$ be finite Abelian groups. Prove that if $G \times H \times H \cong G \times G \times H$, then $G \cong H$.

2. Let $p$ be a prime and $G$ be a group of order $p^n$. For $k \in \{1, 2, 3, \ldots, (n-1)\}$, let $s_k$ and $n_k$ denote the number of subgroups and normal subgroups of $G$ of order $p^k$ respectively. Show that $s_k - n_k$ is divisible by $p$.

Part II

1. Let $R$ be a commutative ring for which every proper ideal is prime. Show that $R$ is a field.

2. Let $F$ be a field and consider the subring $R$ of $F[t]$ given by polynomials with the coefficient of $t$ equal to zero, i.e., $R = F + t^2 F[t]$.
   (a) Show that $R$ has an irreducible element which is not prime. [Hence, $R$ is not PID.]
   (b) Show that $R$ is Noetherian. [Hint: Consider a connection between $R$ and $F[x, y]$.]

Part III

1. Let $R$ be a commutative ring, $S$ be a subring of $R$, $A$ be an $R$-module and
   \[ \mathcal{H} \overset{\text{def}}{=} \text{Hom}_R(R \otimes_S (S \otimes S), A) \].
   Show that for every surjective homomorphism of $R$-modules $\phi : M \to N$ and $R$-module homomorphism $f : \mathcal{H} \to N$ there is an $R$-module homomorphism $F : \mathcal{H} \to M$ such that $\phi \circ F = f$ if and only if the same if true if we replace $\mathcal{H}$ by $A$.

2. Let $R$ be a commutative ring, $D$, $M$ and $N$ be $R$-modules, $\phi : M \to N$ be an $R$-module homomorphism and $1 \otimes \phi : D \otimes_R M \to D \otimes_R N$ be the homomorphism for which
   \[ (1 \otimes \phi)(d \otimes m) = d \otimes \phi(m) \].
   (a) Assume that $\phi$ is injective. Show that if $D$ is free and of finite rank, then $1 \otimes \phi$ is also injective. [The finite rank is not necessary, but we assume it here for simplicity.]
   (b) Show that the above statement is not true for an arbitrary $D$.

Part IV

1. Let $F$ be a field and $K/F$ be an algebraic extension. Show that if $R$ is a subring of $K$ with $F \subseteq R \subseteq K$, then $R$ is a field.

2. Let $F$ be a field, $K/F$ be a Galois extension and $f \in F[x]$ be monic, separable and irreducible. Show that if $f = f_1 \cdots f_k$ is the factorization of $f$ in $K[x]$, with $f_i$ irreducible and monic, then the $f_i$’s are distinct, of the same degree and $G \overset{\text{def}}{=} \text{Gal}(K/F)$ acts transitively on $\{f_1, \ldots, f_k\}$. [I.e., given $\sigma \in G$, the map $f_i \mapsto f_i^\sigma$ is a permutation of the $f_i$’s and given $i, j \in \{1, \ldots, k\}$, there is a $\tau \in G$ such that $f_i^\tau = f_j$.]
ALGEBRA PRELIMINARY EXAMINATION
Spring 2012

Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 455 is necessarily cyclic.

2. Let $G$ be a group of order 56. Show that $G$ is solvable.

Part II.

1. Let $f: \mathbb{Q} \to \mathbb{Z}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$. Show that the image of $f$ has at most three elements and there exist an infinite number of such functions whose image has three elements.

2. Let $R$ be a PID and let $J$ denote the intersection of all maximal ideals of $R$. If $a^2 - a$ is in $J$ for all $a \in R$, then show that $R$ has only finitely many maximal ideals.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and let $M, N$ be projective $R$-modules. Show that $M \otimes_R N$ is a projective $R$-module.

2. Suppose $R$ is a principal ideal domain that is not a field. Suppose $M$ is a finitely generated $R$-module such that for every maximal ideal $P$ of $R$, $M/PM$ is a cyclic $R/P$-module. Show that $M$ itself is cyclic.

Part IV.

1. Let $f(X)$ be a monic polynomial of degree 9 having rational coefficients. Assume that $f(X)$ is irreducible in $\mathbb{Q}[X]$. Let $K$ denote the splitting field of $f$ over $\mathbb{Q}$ and let $u \in K$ be a root of $f$. If $[K : \mathbb{Q}] = 27$, then show that $\mathbb{Q}[u]$ has a subfield $L$ with $[L : \mathbb{Q}] = 3$.

2. Let $F, K$ be fields such that $K$ is a finite Galois extension of $F$ with Galois group $G$. Suppose $a \in K$ is such that $\sigma(a) - a \in F$ for all $\sigma \in G$. If the characteristic of $F$ does not divide the order of $G$, then show that $a \in F$. Assuming $F$ to be the field of two elements, construct a quadratic field extension $K := F[a]$ of $F$ such that $\sigma(a) - a \in F$ for all $\sigma \in G$. 
Algebra Preliminary Exam

January 2011

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send $1$ to $1$, and all $R$-modules are unitary.

I. 1. Let $G$ be a finite simple group. Show that if $G$ has a subgroup $H$ with $[G:H] = n \geq 2$, then $|H| |(n - 1)!$.

2. List, up to isomorphism, all groups of order 153. Justify your answer.

II. 1. Let $R$ be a commutative ring and $I$ an ideal of $R$. Let $I^* = (I, X)$ be an ideal of the polynomial ring $R[X]$. Determine, in terms of $I$, when $I^*$ is a prime ideal of $R[X]$ and when $I^*$ is a maximal ideal of $R[X]$. Justify your answers.

2. (a) Show that if a commutative ring $R$ satisfies DCC on ideals (i.e., $R$ is Artinian), then $R$ has only a finite number of maximal ideals.

(b) Give an example to show that (a) may be false if DCC is replaced by ACC (i.e., if $R$ is Noetherian).
III. 1. Let \( f: M \to M \) be an \( R \)-module homomorphism with \( f \circ f = f \).

Show that the following statements are equivalent.

(a) \( f \) is injective.

(b) \( f \) is surjective.

(c) \( f = 1_M \).

2. (a) Let \( G \) and \( H \) be finitely generated abelian groups such that \( \mathbb{Z}_n \otimes G \cong \mathbb{Z}_n \otimes H \) for every integer \( n \geq 2 \). Show that \( G \cong H \).

(b) Give an example to show that (a) may be false if \( G \) and \( H \) are not both finitely generated.

IV. 1. Let \( F \) be a subfield of a field \( L \). Show that \( L/F \) is an algebraic extension if and only if every subring \( R \) of \( L \) containing \( F \) is a field.

2. Compute the Galois group of \( f(X) = X^4 + X + 1 \in \mathbb{Z}_2[X] \).
Attempt all four parts. Justify your answers.

Part I.

1. How many Sylow 2-subgroups does $S_5$ (the group of permutations of \{1, 2, 3, 4, 5\}) have?

2. Let $G$ be a group of order 231. Show that $G$ is Abelian if and only if $G$ has an Abelian subgroup of order 21.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be a UFD such that each maximal ideal of $R$ is a principal ideal. Prove that $R$ is a PID.

2. Let $\mathbb{R}[[X]]$ denote the power-series ring in a single indeterminate $X$ over the field of real numbers $\mathbb{R}$. If $T$ is a multiplicative subset of $\mathbb{R}[[X]]$ containing 1 but not containing 0, then show that either $T^{-1}\mathbb{R}[[X]] = \mathbb{R}[[X]]$ or $T^{-1}\mathbb{R}[[X]]$ is a field.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and $I$ an ideal of $R$. Show that there exists a surjective $R$-module homomorphism $f : I \to R$ if and only if $I$ is a nonzero principal ideal.

2. Let $K$ be a field, $X$ an indeterminate over $K$ and $M$ a finitely generated $K[X]$-module. Show that $M$ is a projective $K[X]$-module if and only if $M$ is $K[X]$-module isomorphic to $K[X] \otimes_K V$ for some finite dimensional $K$-vector space $V$.

Part IV.

1. Let $K$ be a field and $F$ a subfield of $K$. The group of units of $K$ is denoted by $K^\times$. Suppose $f \in F[X]$ is a monic irreducible polynomial and $a, b \in K^\times$ are such that $f(a) = 0 = f(b)$. Show that the subgroup of $K^\times$ generated by $a$, is isomorphic to the subgroup of $K^\times$ generated by $b$.

2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 4 such that the Galois group of $f$ (over $\mathbb{Q}$) is a group of order 6. Show that $f$ has a root in $\mathbb{Q}$. 
Algebra Preliminary Exam

August 2010

Attempt all problems and justify all answers. All rings have an identity 1 ≠ 0, ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let f : G → H be a surjective homomorphism of finite groups and y ∈ H with |y| = n. Show that there is an x ∈ G with |x| = n.

2. Let p and q be primes, p ≥ q, n ≥ 1, and G a group with |G| = pⁿq. Show that G has a normal subgroup H of order pⁿ. (Hint: do the p > q and p = q cases separately.)

II. 1. Let R be a commutative ring with distinct prime ideals P and Q with P ∩ Q = {0}. Show that R is isomorphic to a subring of the direct product of two fields.

2. Let p and q be positive primes. Show that the polynomial f(X) = X³ + pX² + q ∈ ℤ[X] is irreducible in ℚ[X].

III. 1. Let A and B be finite abelian groups with |A| = m and |B| = n. Show that Homℤ(A, B) = 0 if and only if gcd(m, n) = 1.

2. Let A be a submodule of a projective R-module B. Show that A is projective if B/A is projective.
IV. 1. Let $K \subseteq F$ and $K \subseteq L$ be subfields of a field $M$ with $[F:K] = p$ and $[L:K] = q$ for distinct primes $p$ and $q$. Show that $F \cap L = K$, and that $F = K(\alpha)$ and $L = K(\beta)$ for any $\alpha \in F - K$ and $\beta \in L - K$.

2. Let $K$ be a field and $f(X) \in K[X]$ be irreducible and separable with $\deg(f(X)) = n$. Show that if the Galois group $G$ of $f(X)$ over $K$ is abelian, then $|G| = n$. 