PRACTICE PUTNAM PROBLEMS

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1. FUNCTIONAL RELATIONS

Problem 1 (Mock Putnam Exam UTK 2001). Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(1)| < 1$ and $f(xf(y)) = yf(x)$, for all x, $y \in \mathbb{R}$.

Problem 2 (Greek Math Olympiad 1985). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(f(f(f(x)))) = x$, for all $x \in \mathbb{R}$. Show that there exists $a \in \mathbb{R}$ such that $f(a) = a$.

Problem 3 (Putnam Competition 1990). Determine all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)\ dt\ + 1990,$$

for all $x \in \mathbb{R}$.

Problem 4 (Putnam Competition 1991). Let f and g be non-constant differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $f'(0) = 0$ and

$$f(x + y) = f(x)f(y) - g(x)g(y),$$

$$g(x + y) = f(x)g(y) + g(x)f(y),$$

for all x, $y \in \mathbb{R}$. Show that $(f(x))^2 + (g(x))^2 = 1$, for all $x \in \mathbb{R}$.

Problem 5 (Putnam Competition 1988). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x),$$

for all $x \in \mathbb{R}^+$.

Problem 6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + 2f(2x) + 3f(3x) = 0,$$

for all $x \in \mathbb{R}$.

Problem 7 (International Math Olympiad 1968). Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there exists an $a > 0$ such that

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2},$$

for all $x \in \mathbb{R}$. Show that f is periodic.

Problem 8 (variant, International Math Olympiad 1994). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(y) + xf(y) - x) = f(x) + yf(x) - y,$$

for all x, $y \in \mathbb{R}^+$ and $f(x)/x$ is strictly decreasing on $\mathbb{R}^+$.

Problem 9 (International Math Olympiad 1986). Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(2) = 0,\;\; f(x) \neq 0$ if $0 \leq x < 2$ and

$$f(xf(y))f(y) = f(x+y),$$

for all x, $y \in \mathbb{R}_+$.

Problem 10 (Balkan Math Olympiad 2000). Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = (f(x))^2 + y,$$

for all x, $y \in \mathbb{R}$.

Problem 11 (Putnam Competition 2000). Let $f(x)$ be a continuous function such that

$$f(2x^2 - 1) = 2xf(x),$$

for all $x \in \mathbb{R}$. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Problem 12 (Putnam Competition 1997). Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$, for all x. Prove that $|f(x)|$ is bounded.

Problem 13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is not injective and satisfies

$$f(x+y) = (f(x))^3 + y^3 + f(x)e^y,$$

for all x, $y \in \mathbb{R}$. Show that f is periodic.

Problem 14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|f(x) - f(y)| \;\geq\; \frac{1}{1000}\;|x - y|,$$

for all x, $y \in \mathbb{R}$. Show that f is 1-1 and onto and that f^{-1} is continuous.

Problem 15 (Berkeley Prelim 1999). Let $f : \mathbb{R} \to \mathbb{R}$ be twice-differentiable such that $f(0) = 0$, $f'(0) > 0$ and $f''(x) \geq f(x)$, for all $x \geq 0$. Show that $f(x) > 0$, for all $x > 0$.

Problem 16 (Putnam Competition 1999). Let f be a real-valued function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive and $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Problem 17 (Putnam Competition 1998). Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\; f'(a)\; f''(a)\; f'''(a) \;\geq\; 0.$$

Problem 18. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(nx))^2 + n^2(f(x))^2 \leq 2nf(nx)f(x),$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Problem 19. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(x+1) = f(x+\sqrt{3}),$$

for all $x \in \mathbb{R}$.

Problem 20. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^n f(x)dx = 0,$$

for all $n \in \mathbb{N}$. Show that f is identically zero.

Problem 21 (Berkeley Prelim 1981). Let $f : [1,\infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2},$$

for al $x \geq 1$. Show that $\lim_{x\to\infty} f(x)$ exists and is strictly less that $(\pi + 4)/4$.

Problem 22 (Berkeley Prelim 1996). Let $f : [0,1] \to \mathbb{R}_+$ be a continuous function such that

$$(f(t))^2 \leq 1 + 2\int_0^t f(s)ds,$$

for all $t \in [0,1]$. Show that $f(t) \leq 1 + t$, for all $t \in [0,1]$.

Problem 23 (Asian Pacific Math Olympiad 1989). Find all invertible functions $f : \mathbb{R} \to \mathbb{R}$ which are strictly increasing and satisfy

$$f(x) + f^{-1}(x) = 2x,$$

for all $x \in \mathbb{R}$.

Problem 24 (Asian Pacific Math Olympiad 1994). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$, $f(-1) = -1$, $f(x) \leq f(0)$, for all $x \in (0,1)$ and

$$f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y),$$

for all x, $y \in \mathbb{R}$.

Problem 25 (AM-GM inequality). Let x_1,...,x_n be non-negative reals. Then

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Problem 26. Let n be a positive integer. Show that

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

Problem 27. Let n be a positive integer. Show that

$$1 \cdot 2^2 \cdots n^n < \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}.$$

Problem 28. Find all positive integers a and b such that

$$(a+1)(b+1)(a+b) = 8ab.$$

Problem 29 (Putnam Competition 1998). Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Problem 30 (Asian Pacific Math Olympiad 1995). Determine all sequences of real numbers a_1,...,a_{1995} which satisfy

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1)$$

for $n = 1$, 2, ..., 1994 and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Problem 31 (International Math Olympiad 1984). Let x, y and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$0 \leq xy + yz + zx - 2xyz \leq 7/27.$$

Problem 32 (Asian Pacific Math Olympiad 1996). Let m and n be positive integers such that $n \leq m$. Show that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n.$$

Problem 33 (Cauchy-Schwarz inequality). Let x_1,...,x_n, y_1,...,y_n be real numbers. Then

$$(x_1 y_1 + \cdots x_n y_n)^2 \leq (x_1^2 + \cdots x_n^2)(y_1^2 + \cdots y_n^2)$$

with equality holding if and only if $x_i y_j = x_j y_i$, for all i and j.

Problem 34 (Asian Pacific Math Olympiad 1990). Let a_1,..., a_n be positive real numbers and let S_k be the sum of the products of a_1,...,a_n taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 \cdots a_n$$

for $k = 1$,...,$n - 1$.

Problem 35 (Balkan Math Olympiad 1984). Let a_1,...,a_n be positive real numbers with sum 1. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{1 + \sum_{j \neq i} a_j} \geq \frac{n}{2n-1}.$$

Problem 36 (Balkan Math Olympiad 1985). Let a, b, c, d be real numbers in the interval $[-\pi/2, \pi/2]$ so that

$$sin\ a + sin\ b + sin\ c + sin\ d = 1,$$

$$cos\ 2a + cos\ 2b + cos\ 2c + cos\ 2d \geq 10/3.$$

Show that $0 \leq a,\ b,\ c,\ d \leq \pi/6$.

Problem 37 (Balkan Math Olympiad 2001). Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Show that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

Problem 38 (Putnam Competition 2000). Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1,... are positive reals with $\sum_{j=0}^{\infty} x_j = A$?

Problem 39 (Asian Pacific Math Olympiad 1996). Let a, b, c be the lengths of the sides of a triangle. Show that

$$\sqrt{a+b-c} + \sqrt{a+c-b} + \sqrt{b+c-a} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

and determine when equality occurs.

Problem 40 (International Math Olympiad 1995). Let a, b, c be positive reals with $abc = 1$. Show that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Problem 41 (International Math Olympiad 1999). Let $n \geq 2$ be a fixed integer. Find the smallest possible constant C such that for all non-negative reals x_1,...,x_n we have

$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C(\sum_{i=1}^{n} x_i)^4.$$

Determine when equality occurs.

Problem 42 (International Math Olympiad 2001). Let a, b, c be positive reals. Show that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ac}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Problem 43 (International Math Olympiad 1988). Show that the set of real numbers x satisfying the inequality

$$\frac{1}{x-1} + \frac{2}{x-2} + \cdots + \frac{70}{x-70} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths equals 1988.

Problem 44 (International Math Olympiad 1974). Determine all possible values of

$$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

for positive reals a, b, c, d.

Problem 45 (Asian Pacific Math Olympiad 1991). Let a_1,...,a_n, b_1,...,b_n be positive real numbers such that $a_1 + \cdots a_n = b_1 + \cdots b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \cdots \frac{a_n^2}{a_n+b_n} \geq \frac{a_1 + \cdots a_n}{2}.$$

Problem 46 (Asian Pacific Math Olympiad 1998). Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Problem 47 (Asian Pacific Math Olympiad 1999). Let a_1,...,a_n be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$, for all i, j. Prove that

$$a_1 + \frac{a_2}{2} + \cdots \frac{a_n}{n} \geq a_n,$$

for each positive integer n.

Problem 48. Let α, β, γ be the angles of a triangle. Show that

$$sin(\alpha)\ sin(\beta)\ sin(\gamma) \leq \frac{3\sqrt{3}}{8}.$$

Problem 49. Let $n \geq 2$ and x_1,...,x_n be positive real numbers such that $x_1 + \cdots + x_n = 1$. Show that

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$

Problem 50 (USA Math Olympiad 1980). Let a, b, c be real numbers in $[0,1]$. Show that

$$\frac{a}{b+c+1} + \frac{b}{a+c+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Problem 51 (Weighted power mean inequality). Let x_1,...,x_n, w_1,...,w_n be positive real numbers such that $w_1 + \cdots + w_n = 1$. For a real number $r \neq 0$ define

$$M_w^r(x_1, \cdots, x_n) = (w_1 x_1^r + \cdots + w_n x_n^r)^{1/r}.$$

Also define $M_w^0(x_1, \cdots, x_n) = x_1^{w_1} \cdots x_n^{w_n}$. Then if $r > s$, we have

$$M_w^r(x_1, \cdots, x_n) \geq M_w^s(x_1, \cdots, x_n),$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Problem 52. Let w_1,...,w_n be positive real numbers such that $w_1 + \cdots + w_n = 1$. Show that

$$\sqrt{w_1(1-w_1)} + \cdots + \sqrt{w_n(1-w_n)} \leq \sqrt{n-1}.$$

Problem 53. Minimize the expression

$$x_1 + \frac{x_2^2}{2} + \cdots + \frac{x_n^n}{n}$$

whre x_1,...,x_n are positive real numbers satisfying

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} = n.$$

Problem 54 (Canadian Math Olympiad 1999). Let x, y, z be non-negative reals satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

and determine when equality holds.

Problem 55 (USA Math Olympiad 1979). Let x, y, z be non-negative reals satisfying $x + y + z = 1$. Show that

$$x^3 + y^3 + z^3 + 6xyz \geq \frac{1}{4}.$$

Problem 56 (Hölder's inequality). Let x_1,...,x_n, y_1,...,y_n, α, β be positive reals and suppose that $\alpha + \beta = 1$. Then

$$(x_1 + \cdots + x_n)^\alpha (y_1 + \cdots + y_n)^\beta \geq x_1^\alpha y_1^\beta + \cdots + x_n^\alpha y_n^\beta$$

with equality holding if and only if $x_i y_j = x_j y_i$, for all i, j.

Problem 57 (Minkowski's inequality). Let x_1,...,x_n, y_1,...,y_n, r be positive reals. If $r \geq 1$, then

$$\sqrt[r]{x_1^r + \cdots + x_n^r} + \sqrt[r]{y_1^r + \cdots + y_n^r} \geq \sqrt[r]{(x_1+y_1)^r + \cdots + (x_n+y_n)^r}.$$

If $r < 1$, the inequality is reversed.

Problem 58 (Rearrangement inequality). Let x_1,...,x_n, y_1,...,y_n be real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Then for any permutation σ of $\{1, \cdots, n\}$ we have

$$x_1y_n + x_2y_{n-1} + \cdots + x_ny_1 \leq x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x_ny_{\sigma(n)} \leq x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Problem 59 (Chebychev's inequality). Let x_1,...,x_n, y_1,...,y_n be real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Then

$$\frac{x_1 + \cdots + x_n}{n} \cdot \frac{y_1 + \cdots + y_n}{n} \leq \frac{x_1y_1 + \cdots + x_ny_n}{n}.$$

Problem 60. Let a, b, c, d be positive real numbers. Show that

$$a^a b^b c^c d^d \geq a^b b^c c^d d^a.$$

Problem 61 (Newton's and Maclaurin's inequalities). Let x_1,...,x_n be non-negative reals. For $k \in \{1, \cdots, n\}$, let c_k denote the sum of the products of the x_i taken k at a time. Also let d_k be defined by

$$d_k = \frac{c_k}{\binom{n}{k}}.$$

For convenience, define $c_0 = d_0 = 1$ and $c_k = d_k = 0$ for $k > n$. Then

$$d_k^2 \geq d_{k-1} d_{k+1}, \qquad d_k^{1/k} \geq d_{k+1}^{1/(k+1)}$$

for all $k \in \{1, \cdots, n-1\}$.

Problem 62. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + 1$ be a polynomial with non-negative coefficients. Show that if all the roots of f are real, then $f(x) \geq (1+x)^n$, for all $x \geq 0$.

Problem 63. The real numbers x_1,...,x_n satisfy

$$-1 \leq x_1, \cdots, x_n \leq 1, \qquad x_1^3 + \cdots + x_n^3 = 0.$$

Show that

$$|x_1 + \cdots + x_n| \leq \frac{n}{3}.$$

Problem 64 (Hilbert's theorem, Putnam Competition 1999). Let $p(x)$ be a polynomial with real coefficients such that is $p(x)$ is non-negative for all real x. Prove that, for some k, there are polynomials $f_1(x)$,...,$f_k(x)$ with real coefficients such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Problem 65 (Robinson's example). Consider the polynomial

$$f(x,y) = x^2(x^2 - y^2)(x^2 - 1) + y^2(y^2 - 1)(y^2 - x^2) + (1 - x^2)(1 - y^2).$$

Show that $f(x,y) \geq 0$, for all reals x and y, but $f(x,y)$ cannot be written as the sum of squares of polynomials in x and y with real coefficients.

Problem 66. Find all integers n such that $n^2 + 2n + 6$ is divisible by 25.

Problem 67. Let a, b, c be odd integers. Show that the roots of the equation $ax^2 + bx + c = 0$ are not rational numbers.

Problem 68 (Mock Putnam Exam UTK 2001). Let $f(x)$ be a polynomial with integer coefficients. Suppose there exist distinct integers a, b, c such that $f(a) = f(b) = f(c) = 2000$. Show that there is no integer d such that $f(d) = 2001$.

Problem 69. Let a, b be integers. Show that

$$\left(a + \frac{1}{2}\right)^n + \left(b + \frac{1}{2}\right)^n \in \mathbb{Z}$$

for infinitely many positive integers n if and only if $a + b = -1$.

Problem 70. Find all $n \in \mathbb{Z}$ such that

$$\frac{n^2 + 1}{(n + 1)^2 + 1}$$

is a fraction in lowest terms.

Problem 71. Show that if n is a positive integer, then at least one of the three numbers n, $8n-1$ and $8n + 1$ is composite.

Problem 72. Show that given n integers a_1,...,a_n, there always exists a subset I of $\{1, \cdots, n\}$ such that

$$n \mid \sum_{i \in I} a_i.$$

Problem 73. Given $n + 1$ distinct integers in $\{1, 2, \cdots, 2n\}$, show that we can always choose two of them such that one divides the other. Is the statement true for n integers in $\{1, \cdots, 2n\}$?

Problem 74. Let m, n be positive integers. Show that $4mn - m - n$ can never be a square.

Problem 75 (Putnam Competition 1969). Let n be a positive integer such that $n + 1$ is divisible by 24. Show that the sum of all divisors of n is divisible by 24.

Problem 76 (Asian Pacific Math Olympiad 1998). Show that for any positive integers a and b, the integer $(36a + b)(36b + a)$ cannot be a power of 2.

Problem 77 (Asian Pacific Math Olympiad 1998). Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

Problem 78 (Asian Pacific Math Olympiad 1999). Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Problem 79 (Putnam Competition 1995). The number $d_1d_2...d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2...e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2...d_9$ by the corresponding digit e_i is divisible by 7. The number $f_1f_2...f_9$ is related to $e_1e_2...e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7.

Problem 80 (Putnam Competition 1995). Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - ...}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

Problem 81 (Putnam Competition 1997). Define the sequence $a_1 = 2$, $a_n = 2^{a_{n-1}}$, for $n > 1$. Show that for $n \geq 2$,

$$a_n \equiv a_{n-1} \ (mod\ n).$$

Problem 82 (Putnam Competition 1998). Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = 10$, $A_4 = 101$, and so forth. Determine all n such that 11 divides A_n.

Problem 83 (Putnam Competition 1998). Let N be the positive integer with 1998 decimal units, all of them 1. Find the thousandth digit after the decimal point of $\sqrt{N}$.

Problem 84 (Putnam Competition 1998). Prove that for any integers a, b, c there exists a positive integer n such that

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

Problem 85 (Putnam Competition 1999). Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $(s, n) = 1$ or $(s, n) = s$. Show that there exist s, t in S such that (s, t) is prime.

Problem 86 (Putnam Competition 2000). Prove that there exist infinitely many integers n such that n, $n+1$ and $n+2$ are each the sum of two squares.

Problem 87 (Putnam Competition 2000). Let a_j, b_j and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s and t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j in $\{1, \cdots, N\}$.

Problem 88 (Putnam Competition 2000). Prove that the expression

$$\frac{(m,n)}{n}\binom{n}{m}$$

is an integer for all integers m and n such that $1 \leq m \leq n$.

Problem 89. Find all odd primes p such that $\frac{2^{p-1}-1}{p}$ is a perfect square.

Problem 90. Solve $3^x - 2^y = 1$ in positive integers x and y.

Problem 91 (Balkan Math Olympiad 1998). Show that the equation $y^2 = x^5 - 4$ has no solutions in integers x and y.

Problem 92 (Balkan Math Olympiad 1984). Show that for any positive integer m there exists an integer n such that $n > m$ and the decimal expansion of 5^n is obtained by placing some digits to the left of the decimal expansion of 5^m.

Problem 93 (International Math Olympiad 1968). For every positive integer n, evaluate the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \cdots,$$

where $[x]$ denotes the integral part of x.

Problem 94 (International Math Olympiad 1969). Prove that there are infinitely many positive integers m such that $n^4 + m$ is composite for all integers n.

Problem 95 (International Math Olympiad 1979). Let m and n be positive integers such that

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that m is divisible by 1979.

Problem 96 (International Math Olympiad 1981). Determine the maximum value of $m^2 + n^2$ where m and n are integers in the range 1, 2,...,1981 and $(m^2 - mn - n^2)^2 = 1$.

Problem 97 (International Math Olympiad 1984). Find one pair of positive integers a and b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

Problem 98 (International Math Olympiad 1984). Let a, b, c be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d$ and $b + c$ are both powers of 2, then $a = 1$.

Problem 99 (International Math Olympiad 1985). Let n and k be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, \cdots, n-1\}$ is colored either blue or white. For each i in M, both i and $n - i$ have the same color. For each i in M, with $i \neq k$, both i and $|k - i|$ have the same color. Prove that all elements of M have the same color.

Problem 100 (International Math Olympiad 1985). Given a set M of 1985 positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

Problem 101 (International Math Olympiad 1986). Let d be a positive integer not equal to 2, 5 or 13. Show that one can find distinct a, $b \in \{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Problem 102 (International Math Olympiad 1987). Prove that there does not exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1987$, for all $n \in \mathbb{N}$.

Problem 103 (International Math Olympiad 1989). Prove that for each positive integer n there exist n consecutive positive integers none of which is a prime power.

Problem 104 (International Math Olympiad 1991). Let $n > 6$ be an integer and let a_1, a_2,...,a_k be all the positive integers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0$$

prove that n must be either a prime or a power of 2.

Problem 105 (International Math Olympiad 1994). For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \cdots, 2k\}$ which have exactly three 1s when written in base 2. Prove that for each positive integer m, there is at least one k with $f(k) = m$ and determine all m for which there is exactly one such k.

Problem 106 (International Math Olympiad 1994). Determine all ordered pairs (m, n) of positive integers for which $(n^3 + 1)/(mn - 1)$ is an integer.

Problem 107 (International Math Olympiad 1994). Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist two positive integers m in A and n not in A, each of which is a product of k distinct elements of S for some $k \geq 2$.

Problem 108 (International Math Olympiad 2001). Let n_1,...,n_m be integers with m odd. Let $a = (a_1, \cdots, a_m)$ denote a permutation of the integers 1, 2,...,m. Let $f(a) = a_1 n_1 + \cdots a_m n_m$. Show that there exist distinct permutations a and b such that $f(a) - f(b)$ is a multiple of $m!$.

Problem 109 (International Math Olympiad 2001). Let $a > b > c > d$ be positive integers such that

$$ac + bd = (a + b - c + d)(-a + b + c + d).$$

Show that $ab + cd$ is composite.

4. GEOMETRY

Problem 110 (Mock Putnam Exam, UTK 2001). Show that for any tiling of a square with three triangular tiles one of the tiles must have area equal to half the area of the square.

Problem 111. Show that a square can be dissected into n squares for all $n \geq 6$.

Problem 112 (Putnam Competition 1955). Let $A_1 A_2 \cdots A_n$ be a regular polygon inscribed in a circle of center O and radius R. On the half-line OA_1 choose P such that A_1 is between O and P. Prove that

$$\prod_{i=1}^{n} PA_i = PO^n - R^n.$$

Problem 113 (Balkan Math Olympiad 1984). Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. Let H_i be the orthocenter of $A_kA_lA_m$, where (i, k, l, m) is a permutation of $(1, 2, 3, 4)$. Show that the quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

Problem 114 (Balkan Math Olympiad 2001). Prove that any convex pentagon with equal angles and rational side lengths is a regular pentagon.

Problem 115. Show that there is no regular pentagon whose vertices all have integer coefficients.

Problem 116 (Greek Math Olympiad 1984). Is there a pentagon in 3-space whose angles are all right and whose sides all have equal length?

Problem 117 (International Math Olympiad 1999). Find all finite sets S of at least three points in the plane such that for all distinct points A, B in S, the perpendicular bisector of AB is an axis of symmetry for S.

Problem 118 (Putnam Competition 1997). A rectangle $HOMF$ has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O as the center of the circumscribed circle, M the midpoint of BC and F the foot of the altitude from A. What is the length of BC?

Problem 119 (Putnam Competition 1998). A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem 120 (Putnam Competition 1998). Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

Problem 121 (Putnam Competition 1998). Let A, B, C denote distinct points with integer coordinates in the plane. Prove that if

$$(|AB| + |BC|)^2 < 8|ABC| + 1$$

then A, B, C are three verices of a square.

Problem 122 (Putnam Competition 1998). Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

Problem 123 (Putnam Competition 2000). The octagon $P_1 \cdots P_8$ is inscribed in a circle. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem 124 (Putnam Competition 2000). Three distinct points with integer coordinates lie in the plane on a circle of radius r. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem 125 (Putnam Competition 2000). Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \cdots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Problem 126. Given 69 distinct positive integers not exceeding 100, prove that we can choose four of them a, b, c, d such that $a < b < c$ and $a + b + c = d$.

Problem 127 (German Math Olympiad 1996). Starting at $(1, 1)$, a stone is moved in the plane according to the following rules:
(a) From any point (a, b) the stone can move to $(2a, b)$ or $(a, 2b)$.
(b) From any point (a, b) the stone can move to $(a - b, b)$ if $a > b$ or to $(a, b - a)$ if $a < b$.
For which positive integers x, y can the stone be moved to (x, y)?

Problem 128. Prove that for any prime p, the number $\binom{2p}{p} - 2$ is divisible by p^2.

Problem 129 (Romanian Math Olympiad 1988). Prove that the numbers $\binom{2^n}{k}$ for k in $\{1, 2, \cdots, 2^n - 1\}$ are all even and that exactly one of them is not divisible by 4.

Problem 130 (Balkan Math Olympiad 1985). 1985 people participate in a reunion. In any group of three at least two speak a common language. Knowing that each person at the reunion speaks at most five languages, prove that there exist at least 200 people speaking the same language.

Problem 131 (International Math Olympiad 1986). To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y$, $-y$, $z+y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Problem 132 (International Math Olympiad 1987). Let $p_n(k)$ be the number of permutations of $\{1, 2, \cdots, n\}$ with exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \; p_n(k) = n!$$

Problem 133 (International Math Olympiad 1994). Let m and n be positive integers. Let a_1,...,a_m be distinct elements of $\{1, 2, \cdots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same), we have $a_i + a_j = a_k$ for some k. Prove that

$$\frac{a_1 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

Problem 134 (International Math Olympiad 1998). In a competition there are a contestants and b judges, where $b \geq 3$ and b is odd. Each judge rates each contestant as either "pass" or "fail". Suppose there exists a number k such that for any two judges their ratings coincide for at most k contestants. Prove that $2bk \geq a(b - 1)$.

Problem 135 (International Math Olympiad 2001). 21 girls and 21 boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and for each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least 3 girls and at least 3 boys.

Problem 136 (Putnam Competition 1995). Let S be a set of real numbers which is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three (not necessarily distinct) elements of U is in U, show that at least one of the two sets T, U is closed under multiplication.

Problem 137 (Putnam Competition 1995). Suppose that we have a necklace with n beads. Each bead is labeled with an integer and the sum of all these labels equals $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1,...,x_n satisfy

$$\sum_{i=1}^{k} x_i \leq k-1,$$

for all $k \in \{1, 2, \cdots, n\}$.

Problem 138 (Putnam Competition 1995). For a partition π of $\{1, 2, \cdots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there exist distinct x and y in $\{1, 2, \cdots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

Problem 139 (Putnam Competition 1995). To each positive integer n with exactly n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers (for example, if $n = 2$, there are 9000 such determinants).

Problem 140 (Putnam Competition 1997). Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = h_1 h_2 h_3 = e$. Prove that there exists $a \in G$ such that $\psi(x) = a\phi(x)$ is a group homomorphism.

Problem 141 (Putnam Competition 1997). Let N_n denote the number of ordered n-tuples of positive integers $(a_1, \cdots, a_n)$ such that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 1.$$

Determine whether N_{10} is even or odd.

Problem 142 (Putnam Competition 1999). Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

Problem 143 (Putnam Competition 1999). Evaluate the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n3^m+m3^n)}.$$

Problem 144 (Putnam Competition 1999). Define a sequence by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$ and

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

for $n \geq 4$. Show that a_n is an integer multiple of n, for all n.

Problem 145 (Putnam Competition 1999). Let $A = \{(x, y) : 0 \leq x, y \leq 1\}$. For $(x, y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2}\leq\frac{m}{n}\leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1)}(1 - xy^2)(1 - x^2y)S(x, y)$$

with $(x, y) \in A$.

Problem 146 (Mock Putnam Exam UTK 2001). Evaluate

$$\int_2^{\infty}\sum_{n=1}^{\infty}\left(\frac{(n+1)(n+2)x^2 - 2(n+2)}{2x^2(x^2+1)^n}\right)\,dx.$$

Problem 147 (Putnam Competition 1997). Evaluate

$$\int_0^{\infty}\left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right)\,dx.$$