Algebra diagnostic exam   August 13, 2018   1:30–3:00

In Questions 1,2,3, do not include any mention of matrices in your answers.

All vector spaces are assumed to be finite dimensional.

1. Let $\mathcal{P}_m(\mathbb{R})$ denote the $\mathbb{R}$–vector space consisting of all polynomials in x of degree at most m, with coefficients in $\mathbb{R}$. Suppose that p_0, p_1, ..., p_m are polynomials in $\mathcal{P}_m(\mathbb{R})$ such that each p_j has degree j. Prove that $\{p_0, p_1, \ldots, p_m\}$ is a basis for $\mathcal{P}_m(\mathbb{R})$.

2. Let V, W be complex vector spaces, and let U be a subspace of V. Prove that given a linear map $S : U \to W$, there exists a linear map $T : V \to W$ such that $T(u) = S(u)$ for each $u \in U$.

3. Let $\mathcal{P}_m(\mathbb{C})$ denote the $\mathbb{C}$–vector space consisting of all polynomials in x of degree at most m, with coefficients in $\mathbb{C}$. Suppose that $p_0, p_1, \ldots, p_m$ are elements of $\mathcal{P}_m(\mathbb{C})$ such that $p_j(9) = 0$ $(0 \le j \le m)$. Show that $\{p_0, p_1, \ldots, p_m\}$ is not linearly independent in $\mathcal{P}_m(\mathbb{C})$.

4. Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be a linear map such that 3, 5 and 7 are eigenvalues of T. Suppose also that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^4$. Show that T is invertible.

5. A linear map $T : \mathbb{C}^3 \to \mathbb{C}^3$ is defined by $T(z_1, z_2, z_3) = (0, 2z_1, 3z_2)$. Prove that T does not have a square root, *i.e.* show that there does not exist a linear map $S : \mathbb{C}^3 \to \mathbb{C}^3$ such that $S^2 = T$.

6. Let A be a 4×4 matrix with entries in the complex numbers, satisfying $A^2 \neq 0$, $A^3 = 0$. Determine the Jordan canonical form for A.