Linear Algebra Diagnostic Exam August 12, 2019 1:30–3:00

All vector spaces are assumed to be finite dimensional.

1. Suppose that $\{v_1, \ldots, v_m\}$ is a linearly independent subset of the vector space V, and that $w \in V$. Show that the span of $\{v_1 + w, \ldots, v_m + w\}$ has dimension at least $m - 1$.

2. Let A be a 4×4 matrix with entries in the complex numbers, satisfying $A \neq 0$, $A^2 = 0$. Determine the possible Jordan canonical forms for A.

3. Suppose that U, W are both 4–dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

4. Let $T : V \to V$ be an invertible linear transformation. Prove that a vector $v \in V$ is an eigenvector of T if and only if it is an eigenvector of T^{-1}.

5. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation with eigenvalues 2, 3, 5. Show that there exists a linear transformation $S : \mathbb{C}^3 \to \mathbb{C}^3$ with $S^2 = T$.

6. A linear map $T : \mathbb{C}^3 \to \mathbb{C}^3$ is defined by $T(z_1, z_2, z_3) = (2z_2, 0, 3z_1)$. Prove that T does not have a square root, *i.e.* show that there does not exist a linear map $S : \mathbb{C}^3 \to \mathbb{C}^3$ such that $S^2 = T$.