LINEAR ALGEBRA DIAGNOSTIC TEST–August 10, 2020

All vector spaces assumed to be finite-dimensional, over the complex numbers unless stated otherwise. Justify your answers and include complete proofs for credit.

1. Suppose that V is a 5-dimensional vector space and W is a 3-dimensional vector space. Let $T : V \to W$ be a rank 3 linear transformation. Prove that there exists a 3-dimensional subspace U of V such that the restriction of T to U is invertible.

2. Suppose $V = U \oplus W$ (direct sum), where U, W are subspaces of V, of dimensions respectively 3 and 5. Define a linear transformation $T : V \to V$ by $T(u + w) = u - 2w$ for $u \in U, w \in W$. Find the eigenvalues and corresponding eigenspaces of T, and its minimal and characteristic polynomials.

3. Let T be the linear transformation of $\mathbb{R}^3$ defined by by:

$$T(x) = (2x_2, -x_3, 0), \quad x = (x_1, x_2, x_3).$$

(i) Show that T is nilpotent.

(ii) Let A be the matrix of T in the standard basis of $\mathbb{R}^3$. Compute the matrix exponential $e^{5\mathbb{I}+A}$.

4. Let A be a symmetric real $n \times n$ matrix. Suppose that -3 and 6 are both eigenvalues of A. Prove that there exists a vector $v \in \mathbb{R}^n$ such that $\langle v, Av\rangle = 0$ (for the standard inner product in $\mathbb{R}^n$.)

5. Let $T : V \to V$ be a self-adjoint linear transformation (where V is a real or complex inner product space). Suppose there exists $v \in V$ non-zero so that:

$$||Tv - 3v|| < 9||v||.$$

Prove there exists an eigenvalue λ of T such that $|\lambda - 3| < 9$.

6. Let A be a 4×4 complex matrix whose eigenvalues are $0, 3,$ and 5. Suppose A^2 has rank 3. Determine all possible Jordan canonical forms of A.