LINEAR ALGEBRA DIAGNOSTIC, MAY 2022

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex, unless stated otherwise.

(1) Suppose that V is a 6-dimensional vector space and that W is 4-dimensional vector space. Let $T\colon V \to W$ be a rank 3 linear transformation. Prove that there exists a 3-dimensional subspace U of V such that the restriction of T to U is injective.

(2) Let A be a 4×4 matrix whose eigenvalues are 0, -1, and 3. Suppose that A^2 has rank 3. Find all possible Jordan canonical forms of A, up to reordering the Jordan blocks.

(3) Suppose that -3, 7, 9, and 11 are eigenvalues of a 5×5 matrix A and that A is not diagonalizable. Prove that there exists a $v \in \mathbb{C}^5$ such that

$$Av = \begin{bmatrix} -3 \\ 7 \\ 7i \\ 9 \\ 11 \end{bmatrix}.$$

(4) Suppose that A is a 4×4 Hermitian matrix with eigenvalues -1, 0, 2 and 4. Let v be an eigenvector with eigenvalue 4. If w is a vector orthogonal to v, then show that $\|Aw\| \leq 2\|w\|$.

(5) Let A be a 5×5 Hermitian matrix of rank 3. Show that A^2 also has rank 3.

(6) Let $T\colon V \to V$ be a linear transformation and suppose that there are non-zero (but possibly equal) vectors $v, w \in V$ such that:

$$T(v) = 5w$$
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Show that either 5 or -5 is an eigenvalue of T.