LINEAR ALGEBRA DIAGNOSTIC TEST–January 12, 2021

1. Suppose that V is a 5-dimensional vector space and W is a 3-dimensional vector space. Let $T : V \to W$ be a rank 3 linear transformation. Prove that there exists a 3-dimensional subspace U of V such that the restriction of T to U is invertible.

2. Let $T : \mathbb{C}^8 \to \mathbb{C}^8$ be a linear transformation such that $T^2 = 0$. Prove that the dimension of the image of T is at most 4.

3. Let $P : V \to V$ be a linear map satisfying $P^2 = P$, where V is a real inner product space. Prove that P is an orthogonal projection (that is, the range of P is orthogonal to the nullspace of P) if, and only if, P is self-adjoint.

4. A linear map $T : \mathbb{C}^3 \to \mathbb{C}^3$ is defined by $T(z_1, z_2, z_3) = (0, 4z_1, z_2)$, Prove that T does not have a square root, i.e. show that there does not exist a linear map $S : \mathbb{C}^3 \to \mathbb{C}^3$ such that $S^2 = T$.

5. Let $T : V \to V$ be linear, self-adjoint and positive semi-definite (where V is a real inner product space). Prove that T has a square root. (For the definition of 'square root', see problem 4.)

6. Let A be a 4×4 complex matrix whose eigenvalues are $0, 1,$ and 2. Suppose A^2 has rank 3. Find all possible Jordan canonical forms of A.