LINEAR ALGEBRA DIAGNOSTIC, MAY 2021

All vector spaces are assumed to be finite-dimensional and over the complex numbers, and all matrices have complex entries, unless stated otherwise.

(1) Let U, V, and W be distinct 5-dimensional subspaces of $\mathbb{R}^6$. Prove that there exist 3 linearly independent vectors in $U \cap V \cap W$. Prove that there exist 4 vectors which span $U \cap V \cap W$.

(2) Let A be a 4×4 matrix with minimal polynomial $x^4 - 4x^2$. Find the Jordan canonical form of A.

(3) Let A be a 4×4 matrix of rank 3 such that 1 and 2 are eigenvalues of A (but there may be others). If A is not diagonalizable, find all possible Jordan canonical forms of A (up to reordering the blocks).

(4) Suppose that V is an inner product space with $v, w \in V$. Let $T\colon V \to V$ be the linear operator defined by $T(x) = \langle x, v \rangle w$. Prove that the adjoint of T is defined by $T^*(y) = \langle y, w \rangle v$.

(5) If A is a 5×5 Hermitian matrix and $A^5 = 0$, prove that $A = 0$.

(6) Suppose that A is a Hermitian matrix and $|\lambda| \leq 10$ for every eigenvalue λ of A. If U is a unitary matrix, and λ' is an eigenvalue of AU, prove that $|\lambda'| \leq 10$.