Algebra Diagnostic Exam, May 15, 2019

All vector spaces are assumed to be finite dimensional.

(1) Suppose that W and U are 2-dimensional subspaces of a complex vector space V such that $W \cap U$ is 1-dimensional. Show that there exist vectors $v_1, v_2, v_3 \in V$ such that $\{v_1, v_2\}$ is a basis for W and $\{v_2, v_3\}$ is a basis for U.

(2) Let V and W be complex vector spaces, and suppose that $S\colon V \to W$ and $T\colon V \to W$ are linear transformations. Prove that $\{v \in V : S(v) = T(v)\}$ is a vector subspace of V.

(3) Let W and U be vector subspaces of a vector space V and suppose that the dimension of W equals the dimension of U. Prove that if W and U are distinct, then their union $W \cup U$ is *not* a vector space.

(4) Let $T\colon \mathbb{C}^6 \to \mathbb{C}^6$ be a linear transformation such that $T^2 = 0$. Prove that the dimension of the image of T is at most 3.

(5) Let $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ be a linear transformation and suppose that -1, 3, and 17 are eigenvalues of T. Further suppose that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^4$. Prove that T is invertible.