LINEAR ALGEBRA DIAGNOSTIC, AUGUST 2021

All matrices are assumed to have complex entries, unless stated otherwise.

(1) Let V be an inner product space over the real numbers with vectors v_1, v_2, v_3, and w in V, $w \neq 0$. Suppose that the span of $\{v_1, v_2, v_3\}$ is 2-dimensional and w is orthogonal to v_1, v_2, and v_3. Prove that the span of $\{v_1, v_2, v_3, w\}$ is 3-dimensional.

(2) Suppose that T is a linear transformation on $\mathbb{C}^n$ (for some integer $n \geq 1$) and $v \in \mathbb{C}^n$ is a non-zero vector such that $T(T(v)) = T(v)$. Prove that either 0 or 1 is an eigenvalue of T.

(3) Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be a linear transformation such that $T = T^*$ and the eigenvalues of T are 7, 4, 3, and 1. If v is a vector such that $\|v\| = 2$, then show that $2 \leq \|T(v)\| \leq 14$.

(4) Let A be a 4×4 matrix whose eigenvalues are 2, 3, and 4. If A is not diagonalizable, what are the possible Jordan canonical forms for A?

(5) Let A be a Hermitian (self-adjoint) matrix whose only eigenvalues are -1 and 1. Show that A^2 is the identity matrix.

(6) Suppose that A is a 5×5 matrix such that $A^2 = 0$. Show that the rank of A is at most 2.