ALGEBRA PRELIMINARY EXAM

AUGUST 2023

Instructions: Attempt *all* problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Recall that $C(G)$ denotes the center of a group G.
   (a) Let G be a finite group and let N be a normal subgroup such that $N \subseteq C(G)$ and G/N is cyclic. Show that G is abelian.
   (b) Show that every group of order $255 = 3 \cdot 5 \cdot 17$ is abelian.
2. Let G be a finite p-group and let $C(G)$ denote the center of G. Show that if N is a non-trivial normal subgroup of G then $N \cap C(G)$ is a non-trivial normal subgroup of G.

Part II

1. (a) Show that the polynomial $x + 1$ is a unit in the power series ring $\mathbb{Z}[[x]]$, but is not a unit in $\mathbb{Z}[x]$.
   (b) Show that $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.
2. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal I of $\mathbb{Z}[i]$.

Part III

1. Let R be an integral domain. Prove that R is a field if and only if every R-module is projective.
2. Let R be an integral domain and let Q be its field of fractions. If A is an R-module, prove that every element of $Q \otimes_R A$ can be written as a simple tensor $q \otimes a$ for $q \in Q$ and $a \in A$.

Part IV

1. Let F be a field of prime characteristic p. Suppose $E = F(\alpha)$ is a simple extension such that $\alpha \notin F$ but $\alpha^p - \alpha \in F$.
   (a) Find $[E : F]$.
   (b) Prove that E/F is a Galois extension.
   (c) Find the Galois group $\mathrm{Gal}(E/F)$.
   [Hint: Note that $(x + 1)^p - (x + 1) = x^p - x$ in characteristic p.]
2. Let $\zeta := e^{2\pi i/7}$ be a primitive 7th root of unity and consider the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$.
   (a) Prove that there exists an element $\alpha \in \mathbb{Q}(\zeta)$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.
   (b) Express α explicitly as a polynomial in ζ.

ALGEBRA PRELIMINARY EXAMINATION
SPRING 2023

- **Attempt all four parts. Justify your answers.**
- **<u>Note</u>: Rings are assumed to be commutative and with $1 \neq 0$. Modules are assumed to be unitary left modules. $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{F}_q$ denotes a finite field of q elements.**

Part I.

1. Show that if G is a group of order 2023, then G is an Abelian group.
2. Let G be a group of order 3202 and let $C(G)$ denote the center of G. Show that either G is cyclic or $C(G)$ is trivial. (Hint: 1601 is a prime number.)

Part II.

1. Given Principal Ideal Rings A and B, show that the product-ring $A \times B$ is a Principal Ideal Ring.
2. Suppose n is a positive integer and R is a ring with only n (distinct) maximal ideals such that R_M (= localization of R at the maximal ideal M) is a field for each maximal ideal M of R. Show that there are fields $K_1, \ldots, K_n$ such that R is isomorphic (as a ring) to the product-ring $K_1 \times \cdots \times K_n$.

Part III.

1. Let R be a Principal Ideal Domain and let J be a nonzero proper ideal of R. Suppose n is a positive integer and $h : R^n \longrightarrow \bigoplus_{1 \leq m \leq 2n} R/J^m$ is a R-module homomorphism. Show that h is neither injective nor surjective.
2. Let R be an integral domain with quotient-field K and let M be a R-submodule of K. For an integer $n \geq 2$, suppose the n-fold tensor product $M \otimes_R M \otimes_R \cdots \otimes_R M$ is a torsion-free R-module. Then, given a permutation σ of $\{1, 2, \ldots, n\}$ and $x_1, \ldots, x_n \in M$, show that

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \;=\; x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \qquad (\text{in } M \otimes_R M \otimes_R \cdots \otimes_R M).$$

Part IV.

1. Let $K < L$ be fields such that $[L : K] = 2$. Let E be a purely transcendental field-extension of L of finite transcendence degree. If the fixed-field of $G := Aut(E/K)$ is L, then show that L is purely inseparable over K.
2. Let K be a field and let X be an indeterminate. For an integer n, define

$$f_n \;:=\; X^3 - (4n^2 + 2n + 7)\,X - (4n^2 + 2n + 7) \;\in\; K[X]$$

and let $G(n,\ K)$ denote the Galois-group of f_n over K. For each integer n, determine up to isomorphism, the groups $G(n,\ \mathbb{F}_2)$, $G(n,\ \mathbb{Q})$ and $G(n,\ \mathbb{F}_3)$.

ALGEBRA PRELIMINARY EXAMINATION
FALL 2022

- **Attempt all four parts. Justify your answers.**
- **For a positive integer n, the group of permutations (resp. even permutations) of $\{1,\ldots,n\}$ is denoted by S_n (resp. A_n) and $\mathbb{Z}_n$ denotes the additive group of integers modulo n.**
- **Rings are assumed to be commutative with $1 \neq 0$ and modules are assumed to be unitary left modules.**

Part I.

1. Show that a group of order 81522 is solvable but a group of order $8 \times 15 \times 22$ need not be solvable. (Hint: 647 is a prime divisor of 81522.)
2. If a group G of order 2022 has at least 1 but at most 666 elements of order 6, then show that G is cyclic. (Hint: 337 is a prime divisor of 2022.)

Part II.

1. Let R be a ring and $a,\, b \in R$. For a positive integer n, let $J_n := R\,a^n + R\,b^n$. Show that if J_1 is a principal ideal generated by a non-zerodivisor of R, then J_n is a principal ideal generated by a non-zerodivisor of R for each $n \geq 2$. Find an example of a ring R with elements $a,\, b \in R$ such that for each $n \geq 2$, J_n is a principal ideal generated by a non-zerodivisor of R but J_1 is not a principal ideal of R.
2. Let R be a Unique Factorization Domain. Suppose R has finitely many irreducibles $p_1, \ldots, p_n$ such that each irreducible element of R is an associate of exactly one of $p_1, \ldots, p_n$. Show that R is a Principal Ideal Domain.

Part III.

1. Let R be a Principal Ideal Domain and suppose M is a finitely generated R-module such that $Hom_R(Hom_R(M,\, R),\, R)$ is R-module isomorphic to M. Show that M is a free R-module.
2. Let V be a vector space over $\mathbb{Q}$. For $v_1,\, v_2,\, v_3 \in V$, define

$$f(v_1, v_2, v_3) \;:=\; \sum_{\sigma \in S_3} sgn(\sigma)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \;\in\; V \otimes_{\mathbb{Q}} V \otimes_{\mathbb{Q}} V.$$

Show that $f(v_1, v_2, v_3) = 0$ if and only if $v_1,\, v_2,\, v_3$ are $\mathbb{Q}$-linearly dependent.

Part IV. Let X be an indeterminate.

1. Let $K \leq E$ be fields such that $[E : K] = 2022$ and E/K is Galois. Show the existence of a cubic polynomial $f \in K[X]$ such that f is irreducible in $K[X]$ and has 3 distinct roots in E.
2. Let p be a prime number, let G_p denote the Galois group of $X^6 - p$ over $\mathbb{Q}$ and let

$$\mathfrak{L} \;:=\; \{S_6,\; A_6,\; S_4 \times S_3,\; \mathbb{Z}_{12},\; S_3 \times S_2,\; \mathbb{Z}_6,\; S_3 \times \mathbb{Z}_6,\; A_3 \times \mathbb{Z}_6,\; \mathbb{Z}_2 \times \mathbb{Z}_6\}\,.$$

Determine, with proof, the set of all $H \in \mathfrak{L}$ such that H is isomorphic to G_p for some prime p.

Algebra Preliminary Examination

January 2022

Attempt all questions, and justify each answer.

Part I

1. Let G be a group of order $5175 = 3^2 . 5^2 . 23$. Prove that if H is a normal subgroup of order 23 in G, then H is contained in the center of G.

2. Let G be a group of order $2k$, where k is an odd positive integer. For each element $g \in G$ let σ_g denote the permutation $x \mapsto g\,x$ of G, and let $\Gamma = \{\sigma_g \mid g \in G\}$.

(a) Prove that Γ contains an odd permutation.

(b) Prove that G has a subgroup of order k.

Part II

1. Let R be the ring $\mathbb{Z}[\sqrt{2}]$, consisting of all real numbers $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$. Prove that R is a Euclidean domain, with respect to the norm $N(a + b\sqrt{2}) = |\,a^2 - 2b^2\,|$.

2. Let R be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of R is a prime ideal, then R is a field.

Part III

1. Let R be a commutative ring with $1 \neq 0$. It is assumed that for each ideal I of R the quotient ring R/I is given the natural R–module structure $r\,.\,(x + I) = (rx) + I$.

(a) Let I, J be ideals of R. Prove that $R/I \otimes_R R/J$, $R/(I + J)$ are isomorphic as R–modules.

(b) Let M_1, M_2 be distinct maximal ideals of R. Prove that $R/M_1 \otimes_R R/M_2 = 0$.

2. Let R be the polynomial ring $\mathbb{Z}[x]$, and let $I = (2, x)$, the ideal of R generated by the elements $2, x$. Define R–module homomorphisms $\sigma : R \to R \oplus R$, $\tau : R \oplus R \to I$ as follows:

$$\sigma(h) = (xh\,,\,-2h)\ ,\ \ \tau(f\,,\,g) = 2f + xg\,.$$

(a) Prove that $0 \to R \xrightarrow{\sigma} R \oplus R \xrightarrow{\tau} I \to 0$ is a short exact sequence of R–module homomorphisms.

(b) Prove that I is not a projective R–module.

Part IV

In this part, x denotes an indeterminate.

1. Let $f \in \mathbb{Q}[x]$ be irreducible, with splitting field E over $\mathbb{Q}$. Assume that the degree of E over $\mathbb{Q}$ is an odd integer, and that E contains an intermediate field K with $[K : \mathbb{Q}] = 3$. Prove that the irreducible factors of f, considered as a polynomial over K, all have the same degree.
Hint: First show that K is a normal extension of $\mathbb{Q}$.

2. Let G be the Galois group of the polynomial $f = x^4 + 2x^2 + 4 \in \mathbb{Q}[x]$. Determine the order of G, and describe how each element of G permutes the roots of f.

Algebra Preliminary Examination

August 2021

Attempt all questions, and justify each answer.

Part I

1. Let G be a group. Recall that the *commutator subgroup* $[G\,,\,G]$ of G is the subgroup generated by all commutators $[g_1\,,\,g_2] = g_1^{-1}\,g_2^{-1}\,g_1\,g_2 \quad (g_1, g_2 \in G)$. Also recall that a subgroup H of G is *characteristic in* G , written $H \operatorname{char} G$, if each automorphism of G maps H onto itself.

(a) Define subgroups $G^{(n)}$ $(n \in \mathbb{Z}\,,\, n \geq 0)$ inductively as follows:

$$G^{(0)} = G \quad , \quad G^{(n+1)} = [G^{(n)}\,,\,G^{(n)}] \ .$$

Prove that $G^{(n)} \operatorname{char} G$ for all $n \geq 0$.

(b) Suppose that G is a non-trivial finite group, such that $G^{(n)} = 1$ for some $n > 0$. Prove that G has a non-trivial characteristic subgroup of prime power order. (*Hint:* consider the subgroup $G^{(n-1)}$, where n is the smallest integer for which $G^{(n)} = 1$.)

2. The *holomorph* of a group G, denoted $\operatorname{Hol}(G)$, is defined to be the semidirect product $G \rtimes_\phi \operatorname{Aut}(G)$, where $\phi : \operatorname{Aut}(G) \to \operatorname{Aut}(G)$ is the identity map. Thus we may identify $\operatorname{Aut}(G)$ with the subgroup $K = \{(1\,,\,\sigma) \ : \ \sigma \in \operatorname{Aut}(G)\}$ of the semidirect product $\operatorname{Hol}(G)$. As usual we identify G with the (normal) subgroup $\{(g\,,\,1) \ : \ g \in G\}$ of $\operatorname{Hol}(G)$.

Let $G = \{1\,,\,z_1\,,\,z_2\,,\,z_3\}$ be the non-cyclic group of order 4 (*i.e.* G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). Prove that $\operatorname{Hol}(G)$ is isomorphic to the symmetric group S_4 . (*Hint:* Consider the action by left multiplication of $\operatorname{Hol}(G)$ on the four left cosets $K\,,\, z_1K\,,\, z_2K\,,\, z_3K$ of K .)

Part II

1. Let R be an integral domain with the property that every ideal generated by two elements of R is principal.

(a) Prove that every finitely generated ideal of R is principal.

(b) Suppose that R also satisfies the ascending chain condition on principal ideals, *i.e.* given any chain of principal ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$, there exists a positive integer k such that $I_k = I_{k+n}$ for all positive integers n . Prove that R is a principal ideal domain.

2. Recall that an element e of a ring R is *idempotent* if $e^2 = e$. In this question all rings are assumed to be commutative and with $1 \neq 0$.

(a) Let R be a ring containing an idempotent e distinct from $0\,,\,1$. Prove that R is isomorphic to a direct product of two rings. (*Hint:* if e is idempotent, then so is $1 - e$.)

(b) Suppose that R is a finite ring and that every element of R is idempotent. Prove that R is isomorphic to the direct product of finitely many copies of the field with two elements.

Part III In this part, all *R*–modules M are assumed to be unital, i.e. $1.m = m$ for all $m \in M$.

1. Recall that given left R–modules D, M, N, an R–module homomorphism $\phi : M \to N$ induces a homomorphism of Abelian groups $\phi' : \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N)$ given by $\phi'(\alpha) = \phi \circ \alpha$.

Let R be a ring with $1 \neq 0$ and let D, L, M, N be left R–modules. Prove that if the sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

of module homomorphisms is exact, then the sequence of induced homomorphisms of Abelian groups

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\phi'} \mathrm{Hom}_R(D, M) \xrightarrow{\psi'} \mathrm{Hom}_R(D, N)$$

is also exact.

2. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$, x being an indeterminate. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ inherits from R a natural R–module structure, with annihilator I.

(a) Show that there is an R–module homomorphism from $I \otimes_R I$ to $\mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\dfrac{p(0)}{2} q'(0)$, where q' denotes the usual polynomial derivative of q.

(b) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Part IV In this part, x denotes an indeterminate.

1. This question concerns the polynomial $f(x) := x^{p^n} - x + 1 \in \mathbb{F}_p[x]$ $(n \geq 1)$. We take some fixed algebraic closure $\mathcal{A}$ of $\mathbb{F}_p$, and denote by $\mathbb{F}_{p^k}$ the unique field of order p^k contained in $\mathcal{A}$. You may assume that each extension of finite degree of $\mathbb{F}_p$ is Galois over $\mathbb{F}_p$, with cyclic Galois group generated by the Frobenius automorphism $\phi : a \mapsto a^p$.

(a) Let E be the splitting field over $\mathbb{F}_p$ of $f(x) = x^{p^n} - x + 1$ in $\mathcal{A}$. Show that E contains $\mathbb{F}_{p^n}$ as a subfield. (*Hint:* If α is a root of $f(x)$, then so is $\alpha + a$ for each $a \in \mathbb{F}_{p^n}$.)

(b) Determine the order of the Frobenius automorphism $\phi : E \to E$, $\phi : \beta \mapsto \beta^p$. (*Hint:* First compute $\phi^n(\alpha)$, where α is a root of $f(x)$.)

(c) Show that if $f(x)$ is irreducible over $\mathbb{F}_p$, then $pn = p^n$.
[*Observation (you may omit the easy proof):* from $pn = p^n$ it follows that $n = 1$ or $n = p = 2$.]

2. Determine the Galois group over $\mathbb{Q}$ of $x^4 + 9$, describing how each automorphism permutes the roots of this polynomial.

Algebra Preliminary Examination

January 2021

Attempt all questions, and justify each answer.

Part I

1. Let p be a prime, and let S_p denote the symmetric group of degree p. Prove that if P is a subgroup of S_p of order p, then the normalizer of P in S_p has order $p(p-1)$.

2. Classify, up to isomorphism, the groups of order 63.

Part II

1. A *local ring* is a commutative ring with $1 \neq 0$ that has a unique maximal ideal. Prove that if R is a local ring with maximal ideal M, then every element of $R \setminus M$ is a unit. Also prove that if R is a commutative ring with $1 \neq 0$, in which the set of nonunits forms an ideal M, then R is a local ring with maximal ideal M.

2. Let $p \in \mathbb{Z}_+$ be prime, and let $\mathbb{Z}[i]$ denote the usual ring of Gaussian integers $\{ a+bi \mid a, b \in \mathbb{Z} \}$. For which p is the quotient ring $\mathbb{Z}[i]/(p)$ (i) a field, (ii) a product of fields? Justify your answer. (You may use the following facts: (i) $\mathbb{Z}[i]$ is a Euclidean Domain with respect to the field norm, hence is also a Unique Factorization Domain, and (ii) a prime $p \in \mathbb{Z}_+$ with $p \equiv 1 \pmod 4$ can be written as the sum of two integer squares.)

Hint: Use the Chinese Remainder Theorem where appropriate. Also note that a product of fields cannot contain a nonzero nilpotent element.

Part III

1. Let V be a finite dimensional vector space over a field F, and let v_1, v_2 be nonzero elements of V. Prove that $v_1 \otimes v_2 = v_2 \otimes v_1$ in $V \otimes_F V$ if and only if $v_1 = \lambda v_2$ for some $\lambda \in F$.

2. Let R be a ring with $1 \neq 0$, let P, M, N be R–modules, and let there be an exact sequence of R–module homomorphisms $M \xrightarrow{\phi} N \to 0$.

(a) Prove that if P is a direct summand of a free R–module, then the induced sequence of Abelian group homomorphisms

$$\mathrm{Hom}_R(P, M) \xrightarrow{\phi'} \mathrm{Hom}_R(P, N) \to 0$$

is exact. *(Here ϕ' is the homomorphism $\psi \mapsto \phi \circ \psi$.)*

(b) Show by means of an example that in general the induced sequence $\mathrm{Hom}_R(P, M) \xrightarrow{\phi'} \mathrm{Hom}_R(P, N) \to 0$ need not be exact.

Note: For this question do not assume any result concerning projective modules.

Part IV $\boxed{\textit{In this part, } x \textit{ denotes an indeterminate.}}$

1. This question concerns the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

(a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$, and that its roots in $\mathbb{C}$ are $\pm\alpha$, $\pm\beta$, where $\alpha = \sqrt{1+\sqrt{3}}$, $\beta = \sqrt{1-\sqrt{3}}$.

(b) Prove that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, and that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$.

(c) Prove that the splitting field of $x^4 - 2x^2 - 2$ has degree 8 over $\mathbb{Q}$, and that the Galois group of this polynomial over $\mathbb{Q}$ is dihedral of order 8 .

Hint for (c): The Galois group acts faithfully on the set of roots of the polynomial.

2. Let $\mathbb{F}_p$ denote the field of order p , let $f \in \mathbb{F}_p[x]$ be irreducible over $\mathbb{F}_p$, and let K be a splitting field for f over $\mathbb{F}_p$.

Let L be an intermediate field, *i.e.* $\mathbb{F}_p \subseteq L \subseteq K$. Prove that the irreducible factors of the polynomial f in $L[x]$ all have the same degree.

Hint: Here is one approach. Let $g \in L[x]$ be a factor of f that is irreducible in $L[x]$, and let α be a root of g in K . Consider the relationship between $[L(\alpha) : L]$ and $[K : L]$.

Algebra Preliminary Examination

August 2020

Attempt all questions, and justify each answer.

Part I

1. Let P be a Sylow p–subgroup of a finite group G. If p is the smallest prime dividing $|G|$ and P is cyclic, prove that $N_G(P) = C_G(P)$. (Recall that $N_G(P)$, $C_G(P)$ denote the normalizer and centralizer of P in G, respectively.)

(*Hint:* Consider the order of the automorphism group of P and the action of $N_G(P)$ on P by conjugation.)

2. **(a)** Prove that a group of order 105 contains a cyclic normal subgroup of order 35.

(b) Prove that, up to isomorphism, there is just one non-Abelian group of order 105.

In parts II, III and IV, X denotes an indeterminate.

Part II

1. Let R be a commutative ring with $1 \neq 0$. Recall that R is *Artinian* if it satisfies the descending chain condition on ideals, *i.e.* if $I_1 \supseteq I_2 \supseteq \dots$ is a descending chain of ideals of R, then there exists $k \in \mathbb{Z}_+$ such that $I_m = I_k$ for all $m > k$.

Let S be an arbitrary commutative ring with $1 \neq 0$, and let J denote the Jacobson radical of $S[X]$. Prove that $S[X]/J$ is not Artinian.

2. Let R be the subring of $\mathbb{Q}[X]$ consisting of all polynomials whose constant term is an integer.

(a) Prove that R is an integral domain in which every irreducible element is prime.

(b) Prove that R is not a Unique Factorization Domain.
(*Hint:* Consider factorizations of the element X.)

Part III

1. Let k be a field, and let $R = M_2(k)$ be the ring of 2×2 matrices over k. Let P be the set of 2×1 matrices over k: then P is an Abelian group under matrix addition, and left matrix multiplication of elements of P by elements of R accords P the structure of a left R–module.

Prove that the R–module P is projective, but not free.

2. Let $R = \mathbb{Z}[X]$, let $I \subset R$ be the ideal generated by $2, X$, and let $M = I \otimes_R I$.

Prove that the element $2 \otimes 2 + X \otimes X \in M$ cannot be written as a simple tensor $a \otimes b$ $(a, b \in I)$.

(*Hint:* Use a suitable R–module homomorphism defined on M.)

Part IV

1. Prove that $\mathbb{Q}(\sqrt{5+2\sqrt{5}})$ is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

2. Let F be a field (possibly infinite) of finite characteristic p, and let $a \in F$ be an element not of form $b^p - b$ for any $b \in F$. Let $f = X^p - X - a \in F[X]$.

(a) Prove that the polynomial f is separable and irreducible over F.

(b) Prove that if α is a root of f in an extension field of F, then $F(\alpha)$ is a splitting field for f over F.

(*Hint:* Consider the effect of substituting $X+1$ for X in the polynomial f.)

ALGEBRA PRELIMINARY EXAM

JANUARY 2020

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a finite group and $\phi : G \to H$ a *surjective* homomorphism. Prove that if $y \in H$ is such that $|y| = p^r$, for some prime p and $r \in \mathbb{Z}_{>0}$, then there is $x \in G$ such that $\phi(x) = y$ and $|x| = p^s$, for some $s \in \mathbb{Z}_{>0}$.

[**Hint:** Let $g \in G$ such that $\phi(g) = y$, and write $|g| = n \cdot p^k$, where $p \nmid n$.]

2. Let G be a group of order 60 and assume that 4 divides $|Z(G)|$ [where $Z(G)$ denotes the *center* of G]. Prove that G must be Abelian.

Part II

1. Let I be the ideal of $\mathbb{Z}[x]$ generated by 7 and $x^2 + 1$. Prove that I is a maximal ideal.

2. Let R be an *integral domain* such that for any descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

there is a positive integer N such that $I_i = I_N$ for all $i \geq N$. Prove that R is a field.

Part III

1. Let R be a subring of S. Prove that $S \otimes_R S \neq 0$.

2. Let R be a ring containing $\mathbb{Z}$ such that R is a free $\mathbb{Z}$-module of finite rank $n > 0$ and every non-zero ideal of R has a non-zero element of $\mathbb{Z}$. Prove that for every non-zero ideal I we have that R/I is finite.

Part IV

1. Given a prime p and a positive integer n, show that there is an *Abelian* extension [i.e., Galois with Abelian Galois group] K of $\mathbb{Q}$ with $[K : \mathbb{Q}] = p^n$.

2. Let F be a field of characteristic p with exactly p^r elements. If K is a finite extension of F with $K = F[\alpha]$, for some $\alpha \in K$, and f is the minimal polynomial of α over F, then show that if β is another root of f, then $\beta \in K$ and $\beta = \alpha^{p^k}$ for some $k \in \mathbb{Z}$.

ALGEBRA PRELIMINARY EXAM

AUGUST 2019

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G_1, G_2 be groups, $N \trianglelefteq G_1$, and $\phi : G_1 \to G_2$ be an onto homomorphism such that $N \cap \ker(\phi) = \{1\}$. Prove that for $x \in N$ we have that $\mathrm{C}_{G_2}(\phi(x)) = \phi(\mathrm{C}_{G_1}(x))$. [Remember: $\mathrm{C}_G(x) \stackrel{\text{def}}{=} \{g \in G : gx = xg\}$ is the *centralizer* of x in G.]

2. Let G be a group of order $992 = 2^5 \cdot 31$. Prove that either G has a normal subgroup of order $32 = 2^5$ or it has a normal subgroup of order 62.

Part II

1. Let R be a UFD with exactly two non-associate prime elements p and q [i.e., p and q are non-associate primes and every prime is an associate of either p or q]. Prove that R is a PID.

2. Let R be a PID and P a prime ideal of $R[x]$ such that $P \cap R \neq \{0\}$. Prove that there is $p \in R$ prime [in R] such that either $P = (p)$ or $P = (p, f)$ for some f prime in $R[x]$.

Part III

1. Let R be a commutative ring and M an R-module. Prove that $R \otimes_R \mathrm{Hom}_R(R \oplus R, M)$ is projective if and only if M is projective.

2. Let R be a commutative ring, M and N be R-modules and M' and N' be submodules of M and N respectively. Define L as the submodule of $M \otimes_R N$ generated by the set
$$\{x \otimes y \in M \otimes_R N \ : \ x \in M' \text{ or } y \in N'\}.$$
Show that $M/M' \otimes_R N/N' \cong (M \otimes_R N)/L$.

Part IV

1. Let $F = \mathbb{Q}(\sqrt[3]{2} \cdot \zeta)$, where $\zeta = -1/2 + \sqrt{3}\mathrm{i}/2$ [a primitive third root of unity]. Prove that -1 is not a sum of squares in F, i.e., there is no positive integer n and $\alpha_1, \ldots, \alpha_n \in F$ such that $-1 = \alpha_1^2 + \cdots + \alpha_n^2$.

2. Let F be a field of characteristic 0 and K/F be a field extension of degree n such that there is a root of unity ζ in the algebraic closure of K such that $K \subseteq F[\zeta]$. Prove that if $d \mid n$, there is $\alpha \in K$ such that the minimal polynomial of α over F has degree d.

ALGEBRA PRELIMINARY EXAM

AUGUST 2018

Instructions: Attempt *all* problems in all four parts. Justify your answers.

General assumptions: All rings have $1 \neq 0$, their subrings contain 1, and all modules are unitary.

Part I

1. Let G be a (possibly infinite) group, and suppose that G contains a subgroup $H \neq G$ whose index $[G : H]$ is finite. Prove that G contains a normal subgroup $N \neq G$ of finite index.
2. Prove that every group of order 70 contains a cyclic, normal subgroup of order 35.

Part II

1. Let R be a commutative ring in which every element is either a unit or nilpotent. Prove that R has exactly one prime ideal.
2. If R is an integral domain, prove that there are infinitely many ideals in $R[x]$ that are both prime and principal.

Part III

1. Let R be a ring, possibly non-commutative, and suppose that
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence of left R-modules, with M' and M'' finitely generated. Prove that M is finitely generated.
2. Let M be a finitely-generated $\mathbb{Z}$-module, and let $T \subset M$ be its torsion submodule. Show that the torsion submodule of $M \otimes_{\mathbb{Z}} M$ has at least $|T|$ elements.

Part IV

1. Let p be a prime and suppose that $f \in \mathbb{F}_p[x]$ is an irreducible polynomial. Let K be a degree 2 extension of $\mathbb{F}_p$ and suppose that there exist non-constant polynomials $g, h \in K[x]$ such that $f = gh$. If g is an irreducible polynomial of degree 5, what is the degree of f?
2. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial, and $K/\mathbb{Q}$ is an extension such that f has exactly one root in K. Let G be the Galois group of f, and show that $|G|$ is divisible by 12.

ALGEBRA PRELIMINARY EXAM

AUGUST 2017

Instructions: Attempt *all* problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Suppose that H is a subgroup of a finite group G of index p, where p is the smallest prime dividing the order of G. Prove that H is normal in G.
2. Show that every group of order 222 is solvable.
   Fun fact: The University of Tennessee was established 222 years ago.

Part II

1. Let I and J be ideals of a ring R and assume that P is a prime ideal of R that contains $I \cap J$. Prove that either I or J is contained in P.
2. Let R be an integral domain and suppose that every prime ideal in R is principal. Prove that R is a PID.

Part III

1. Let V be a Noetherian right R-module, and $\theta : V \to V$ a homomorphism.
   (a) Show that $\ker(\theta^{n+1}) = \ker(\theta^n)$ for some $n \geq 1$.
   (b) If θ is onto, show that it is one-to-one.
2. An R-**projection** is defined to be an R-module homomorphism $\varphi : R^n \to R^n$ such that $\varphi^2 = \varphi$. Prove that a finitely generated R-module M is projective if and only if it is isomorphic to the image of some R-projection.

Part IV

1. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F(\alpha)$. Assume that some power of α lies in F and let n be the smallest positive integer such that $\alpha^n \in F$.
   (a) If $\alpha^m \in F$ with $m > 0$, show that m is a multiple of n.
   (b) If E is a separable extension of F, prove that the characteristic of F does not divide n.
   (c) If every root of unity of E lies in F, show that $[E : F] = n$.
2. Let F be a field of characteristic 0 and let E be a finite Galois extension of F.
   (a) If $0 \neq \alpha \in E$ with $E = F(\alpha)$, show that $F(\alpha^2) \neq E$ if and only if there exists $\sigma \in \mathrm{Gal}(E/F)$ with $\sigma(\alpha) = -\alpha$.
   (b) Prove that there exists an element $\alpha \in E$ with $E = F(\alpha^2)$.

ALGEBRA PRELIMINARY EXAMINATION
SPRING 2017

- **Attempt all four parts. Justify your answers.**

Part I.

1. Show that a group of order 255 is not a simple group.
2. A group G has a cyclic normal subgroup of order 2016. If G also has a subgroup of order 2017, then show that G has a cyclic subgroup of order $(2016) \times (2017)$.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$.

1. Let A and B be rings. Show that each ideal of $A \times B$ is of the form $I \times J$, where I is an ideal of A and J is an ideal of B.
2. Let R be a ring, let X be an indeterminate and let $S := \{X^n \mid 0 \leq n \in \mathbb{Z}\}$. If $S^{-1}R[[X]]$ is a field, then show that R is a field.

Part III.

Note : *Rings are assumed to be commutative with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let A be a ring and let M, N be finitely generated projective (left) A-modules. Show that $Hom_A(M, N)$ is a finitely generated projective A-module.
2. Let R be a PID and let I, J be ideals of R. If $I \neq R \neq J$, then show that $(R/I) \oplus (R/J)$ and $(R/I) \otimes_R (R/J)$ are not isomorphic as (left) R-modules.

Part IV.

Note : *In what follows, X is an indeterminate.*

1. Let K be an extension-field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}_{30}$. Suppose each of $f, g \in \mathbb{Q}[X]$ is an irreducible polynomial of degree 6 and f has a root $a \in K$. If g has a root in K, then show that g has all its roots in $\mathbb{Q}[a]$.
2. Let $F \subset K$ be finite fields of characteristic 5 and suppose $g \in F[x]$ is irreducible in $F[x]$. If g has degree 11, then show that either g is irreducible in $K[x]$ or all its roots are in K.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2016

- **Attempt all four parts. Justify your answers.**

Part I.

1. Let p be a prime number and G be a non-Abelian group of order p^3. Show that G has at least 3 (distinct) subgroups of index p.
2. Let G be a group of order p^3q, where p, q are distinct prime numbers. If no Sylow p-subgroup of G is normal and also no Sylow q-subgroup of G is normal, then show that G has order 24.

Part II.

Note : *Rings are tacitly assumed to be commutative and with* $1 \neq 0$.

1. Let R be a ring, X an indeterminate and $h : R[X] \to R[[X]]$ a ring-homomorphism such that $h(a) = a$ for all $a \in R$. Show that h is not surjective.
2. Let R be an integral domain with at least 3 (distinct) maximal ideals. Given maximal ideals M and N of R, show that $R_M \cap R_N \neq R$. (Here localization of R at a prime ideal is naturally identified as a ring in between R and the quotient-field of R.)

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be a ring and let $a \in R$ be a nonzero element of R such that $a^3 = a$. Show that the ideal Ra is a projective R-module.
2. Let R be a PID and let M be a finitely generated R-module. For a maximal ideal Q of R, let $\delta(Q, M)$ denote the dimension of $M \otimes_R R/Q$ as a vector-space over the field R/Q. Let $\delta(M)$ denote the $\sup\{\delta(Q, M)\}$, where the supremum is taken over all maximal ideals Q of R. Show that as an R-module, M has a generating set of cardinality $\delta(M)$ and any generating set of M has cardinality at least $\delta(M)$.

Part IV.

Note : *In what follows,* X *is an indeterminate.*

1. Let $f(X)$ be a monic polynomial with rational coefficients. Assume $f(X)$ is irreducible in $\mathbb{Q}[X]$ and the Galois-group of $f(X)$ over $\mathbb{Q}$ is a group of order 99. What is the degree of $f(X)$?
2. Compute the Galois group of $X^6 - 9$ over $\mathbb{Q}$.

ALGEBRA PRELIMINARY EXAM

JANUARY 2016

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a finite group and H be a subgroup of G. Prove that $n_p(H) \leq n_p(G)$, where $n_p(X)$ denotes the number of Sylow p-subgroups of X.

2. Let G be a group of order p^n for some prime p and positive integer n. Prove that if $1 \neq H \trianglelefteq G$, then $Z(G) \cap H \neq 1$. [Here $Z(G)$ denotes the center of G.]

Part II

1. Let R be a *Boolean ring*, i.e., a ring [with 1] for which $a^2 = a$ for all $a \in R$. [You can use without proof the well known fact that if R is Boolean, then it is commutative of characteristic 2.]

(a) Prove that if R is finite, then its order is a power of 2.
(b) Prove that every prime ideal of R is maximal.

2. Show that $R \stackrel{\text{def}}{=} \mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1x_2, x_3x_4, x_5x_6, \ldots)$ has infinitely many distinct *minimal* prime ideals. [P is a minimal prime ideal if it is prime and whenever $Q \subseteq P$, with Q also prime, we have $Q = P$.]

Part III

1. Let F be a field and M be a *torsion* $F[x]$-modulo. Prove that if there is $m_0 \in M$, with $m_0 \neq 0$, and an *irreducible* $f \in F[x]$ such that $f \cdot m_0 = 0$, then $\mathrm{Ann}(M) \subseteq (f)$.

2. Let R be an integral domain and I a principal ideal of R. Prove that $I \otimes_R I$ has no non-zero torsion element [i.e., if $m \in I \otimes_R I$, with $m \neq 0$, and $r \in R$ with $rm = 0$, then $r = 0$].

Part IV

1. Let K/F be an algebraic field extension and $\mathrm{Emb}(K/F)$ denote the set of field homomorphisms $\sigma : K \to \bar{K}$ such that $\sigma(a) = a$ for all $a \in F$. [Here $\bar{K}$ is a fixed algebraic closure of K.]

(a) Prove that if α is a root of a [not necessarily irreducible] non-zero polynomial $f \in F[x]$ with $\deg(f) = n$, then $\mathrm{Emb}(F[\alpha]/F)$ has at most n elements.
(b) Give an example of an algebraic extension K/F of degree greater than one for which $\mathrm{Emb}(K/F)$ has a single element.

2. Let $F = \mathbb{Q}[\sqrt{2}]$ and $K = \mathbb{Q}[\sqrt[8]{2}, \mathrm{i}]$.

(a) Prove that K/F is Galois with $[K : F] = 8$.
(b) Prove that $\mathrm{Gal}(K/F)$ has a non-normal subgroup. [This implies that it is the dihedral group of order 8, as it is the only group of order 8 with this property.]

ALGEBRA PRELIMINARY EXAM

AUGUST 2015

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a *non-Abelian* group of order p^3, $[G,G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle$ be its commutator subgroup and $Z(G)$ be its center. Show that $|Z(G)| = p$ and that $Z(G) = [G,G]$.

2. Let G_1 and G_2 be groups of order 81 acting *faithfully* [i.e., only 1 acts as the identity function] on sets X_1 and X_2, respectively, with 9 elements each. Show that there is an isomorphism $\psi : G_1 \to G_2$.

Part II

1. Let D be a *finite* division ring. Prove that D has a prime power number of elements. [**Hint:** Consider the center $Z(D) = \{a \in D : ax = xa \text{ for all } x \in D\}$.]

2. Let $p \in \mathbb{Z}$ prime and

$$f = a_{2n+1}x^{2n+1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x].$$

Prove that if $p^3 \nmid a_0$, $p^2 \mid a_0, a_1, \ldots, a_n$, $p \mid a_{n+1}, a_{n+2}, \ldots, a_{2n}$ and $p \nmid a_{2n+1}$, then f is irreducible in $\mathbb{Q}[x]$.

Part III

1. Let R be a commutative ring. An R-module is *Artinian* if it satisfies the *descending chain condition for submodules.* [I.e., if $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ is a chain of submodules, then there is a i_0 such that for all $i \geq i_0$, we have $S_i = S_{i_0}$.] Show that if L and N are Artinian R-modules and we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0,$$

then M is also Artinian.

2. Let R be a commutative ring such that every R-module is free. Prove that R is a field.

Part IV

1. Let $\mathbb{F}_p$ be the field with p elements, and t be an indeterminate. Let $f(t), g(t) \in \mathbb{F}_p[t] \setminus \{0\}$, with $\max\{\deg f, \deg g\} < p$ and $f(t)/g(t) \notin \mathbb{F}_p$. Show that the extension $\mathbb{F}_p(t)/\mathbb{F}_p(f(t)/g(t))$ is separable.

2. Suppose that $f = \prod_{i=1}^{N}(x - \alpha_i) \in \mathbb{Q}[x]$ [with $\alpha_i \in \mathbb{C}$] is *irreducible* in $\mathbb{Q}[x]$ and let $f_n \stackrel{\text{def}}{=} \prod_{i=1}^{N}(x - \alpha_i^n)$. Prove that for each n, there is $g_n \in \mathbb{Q}[x]$ *irreducible* and a positive integer k_n such that $f_n = g_n^{k_n}$.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2014

Attempt all four parts. Justify your answers.

Part I.

1. Show that S_4 (the group of permutations of $\{1, 2, 3, 4\}$) does not have a subgroup isomorphic to Q_8 (the quaternion-group of order 8).

2. Let G be a group of order 2014. Show that G is cyclic if and only if G has a normal subgroup of order 2.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$*, subrings are assumed to contain* 1 *and ring-homomorphisms are assumed to map* 1 *to* 1.

1. Let R be an integral domain with only finitely many units. Show that the intersection of all maximal ideals of R is 0.

2. Let R be a ring such that each non-unit of R is nilpotent. Let X be an indeterminate and let $f \in R[[X]]$. Show that $f^n = f$ for some integer $n \geq 2$ if and only if either $f = 0$ or $f^{n-1} = 1$.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let L be a module over a ring R and let M, N be R-submodules of L. Show that if $(M + N)/(M \cap N)$ is a projective R-module then $M/(M \cap N)$ is also a projective R-module.

2. Let R be a PID with infinitely many prime ideals and let M be a finitely generated R-module. Show that M is a torsion R-module if and only if $M \otimes_R R/P = 0$ for all but finitely many prime ideals P of R.

Part IV.

Note : *In what follows,* X *is.an indeterminate.*

1. Let $f(X) := X^5 + 3X^3 + X^2 + 3 \in \mathbb{Q}[X]$. Let K be the splitting field of $f(X)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.

2. Let $f(X) := X^3 + X + 1 \in \mathbb{Q}[X]$. Let F be a finite Galois extension of $\mathbb{Q}$ such that the Galois group of F over $\mathbb{Q}$ is an Abelian group. Show that f is irreducible in $F[X]$.

Algebra Preliminary Exam January 2014

Attempt all problems and justify all your answers. All rings have a $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I. Groups

1. Show that every group of order 1,225 is abelian.
2. Let $n \geqslant 2$. Show that there is a nontrivial homomorphism $f : S_n \to \mathbb{Z}/n\mathbb{Z}$ (i.e., $\ker f \neq S_n$) if and only if n is even.

Part II. Rings

1. Let R be a commutative ring. Show that $J(R[X]) = \text{nil}(R[X])$. ($J(A)$ and $\text{nil}(A)$ are the Jacobson and nil radicals of A.)
2. Let R be a PID.
   (a) Show that R satisfies ACC on ideals.
   (b) Show that every nonzero prime ideal of R is maximal.

Part III. Modules

1. Let R be a ring and M a nonzero R-module. Show that $M = A \oplus B$ for proper submodules A and B of M if and only if there is a nonzero, nonidentity homomorphism $f : M \to M$ with $f^2 = f$.
2. Let R be a commutative ring, I a proper ideal of R, and M an R-module. Show that $(R/I) \otimes_R M$ and M/IM are isomorphic as R-modules.

Part IV. Fields

1. Let K a subfield of a field F. Show that there is a subring of F containing K that is a PID, but not a field, if and only if the extension F/K is not algebraic.
2. Determine the Galois group of $f(X) = X^{10} + X^8 + X^6 + X^2$ over $\mathbb{Z}/2\mathbb{Z}$.

Algebra Preliminary Exam

August 2013

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I.

1. (a) Let p and q be (not necessarily distinct) prime numbers. Show that a group G with $|G| = pq$ is either abelian or $Z(G) = \{e\}$.

   (b) Give an example of a nonabelian group G whose order is the product of three (not necessarily distinct) primes and $Z(G) \neq \{e\}$.

2. (a) Let G be a group with $|G| = 100$. Show that G is abelian if and only if its Sylow 2-subgroup is normal.

   (b) Give an example of a nonabelian group of order 100.

Part II.

1. Let R and S be a commutative rings with $1 \neq 0$. Show that every ideal of $R \times S$ has the form $I \times J$ for I an ideal of R and J an ideal of S.

2. Let R be a commutative ring with $1 \neq 0$. Show that $f(X) = a_0 + a_1X + \cdots + a_nX^n$ is a unit in $R[X]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent.

Part III

1. Let P and Q be finitely generated projective R-modules over a commutative ring R with $1 \neq 0$. Show that $\mathrm{Hom}_R(P,Q)$ is a finitely generated projective R-module.

2. Let R be a commutative ring with $1 \neq 0$, S a nonempty multiplicatively closed subset of R, and M an R-module. Show that $(S^{-1}R)\otimes_R M$ and $S^{-1}M$ are isomorphic as $S^{-1}R$-modules.

Part IV.

1. Let p and q be distinct prime numbers, F a subfield of a field K, and $f(X)$, $g(X) \in F[X]$ be irreducible with $\deg(f(X)) = p$ and $\deg(g(X)) = q$. Let $a, b \in K$ be roots of $f(x)$ and $g(X)$, respectively. Show that $[F(a,b):F] = pq$.

2. (a) Let F be a splitting field for $f(X) \in \mathbb{Q}[X]$ over $\mathbb{Q}$ with abelian Galois group G. Show that every subfield L of F is a splitting field over $\mathbb{Q}$ for some polynomial $g(X) \in \mathbb{Q}[X]$.

   (b) Give an example to show that if G is not abelian in part (a), then some L need not be a splitting field.

ALGEBRA PRELIMINARY EXAM

JANUARY 2013

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let p and q be prime numbers such that $q < p$ and q does not divide $p^2 - 1$. Prove that every group of order p^2q is Abelian.

2. Let G be a finite *simple* group. Show that if p is the *largest* prime dividing $|G|$, then there is no subgroup $H \leq G$ such that $1 < |G : H| < p$.

Part II

1. Let R be a ring not necessarily having 1 [or commutative], with at least two elements and such that for all non-zero $a \in R$ there is a *unique* $b \in R$ such that $aba = a$.

(a) Show that R has no [non-zero] zero divisors.
(b) Show that for a and b as above, we also have $bab = b$.
(c) Show that R has 1.

2. Let R be a commutative ring and $a \in R$ such that $a^n \neq 0$ for all positive integers n. Let I be an ideal maximal with respect to the property that $a^n \notin I$ for any positive integer n. Show that I is prime.

Part III

1. Let $V = \mathbb{R}^2$ and $\{e_1, e_2\}$ be a basis of V. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_{\mathbb{R}} V$ cannot be written as a single tensor.

2. Let R be a PID.

(a) Prove that a finitely generated R-module M is free if and only if it is torsion free.
(b) Prove that if a finitely generated R-module M is projective, then it is free.

Part IV

1. Let $\mathbb{F}_p$ be the field with p elements, $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of $\mathbb{F}_p$ and let

$$L = \{\alpha \in \bar{\mathbb{F}}_p \ : \ p \nmid [\mathbb{F}_p[\alpha] : \mathbb{F}_p]\}.$$

Show that L is a field.

2. Let p be a prime, F be a field of characteristic different from p and $f = x^p - a \in F[x]$ [not necessarily irreducible]. Let K be the splitting field of $x^p - 1$ over F and assume that all roots of f lie in K.

(a) Show that if $f(\alpha) = 0$ with $\alpha \notin F$, then $F[\alpha] = K$.
(b) Prove that f has a root in F.

ALGEBRA PRELIMINARY EXAM

AUGUST 2012

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: All rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G and H be finite Abelian groups. Prove that if $G \times H \times H \cong G \times G \times H$, then $G \cong H$.

2. Let p be a prime and G be a group of order p^n. For $k \in \{1, 2, 3, \ldots, (n-1)\}$, let s_k and n_k denote the number of subgroups and normal subgroups of G of order p^k respectively. Show that $s_k - n_k$ is divisible by p.

Part II

1. Let R be a commutative ring for which every proper ideal is prime. Show that R is a field.

2. Let F be a field and consider the subring R of $F[t]$ given by polynomials with the coefficient of t equal to zero, i.e., $R = F + t^2F[t]$.

(a) Show that R has an irreducible element which is not prime. [Hence, R is not PID.]

(b) Show that R is Noetherian. [**Hint:** Consider a connection between R and $F[x, y]$.]

Part III

1. Let R be a commutative ring, S be a subring of R, A be an R-module and

$$\mathcal{H} \stackrel{\text{def}}{=} \operatorname{Hom}_R(R \otimes_S (S \oplus S), A).$$

Show that for every *surjective* homomorphism of R-modules $\phi : M \to N$ and R-module homomorphism $f : \mathcal{H} \to N$ there is an R-module homomorphism $F : \mathcal{H} \to M$ such that $\phi \circ F = f$ if and only if the same if true if we replace $\mathcal{H}$ by A.

2. Let R be a commutative ring, D, M and N be R-modules, $\phi : M \to N$ be an R-module homomorphism and $1 \otimes \phi : D \otimes_R M \to D \otimes_R N$ be the homomorphism for which

$$(1 \otimes \phi)(d \otimes m) = d \otimes \phi(m).$$

(a) Assume that ϕ is injective. Show that if D is free and of finite rank, then $1 \otimes \phi$ is also injective. [The finite rank is not necessary, but we assume it here for simplicity.]

(b) Show that the above statement is not true for an arbitrary D.

Part IV

1. Let F be a field and K/F be an algebraic extension. Show that if R is a *subring* of K with $F \subseteq R \subseteq K$, then R is a field.

2. Let F be a field, K/F be a Galois extension and $f \in F[x]$ be monic, separable and irreducible. Show that if $f = f_1 \cdots f_k$ is the factorization of f in $K[x]$, with f_i irreducible and monic, then the f_i's are distinct, of the same degree and $G \stackrel{\text{def}}{=} \operatorname{Gal}(K/F)$ acts *transitively* on $\{f_1, \ldots, f_k\}$. [I.e., given $\sigma \in G$, the map $f_i \mapsto f_i^\sigma$ is a permutation of the f_i's and given $i, j \in \{1, \ldots, k\}$, there is a $\tau \in G$ such that $f_i^\tau = f_j$.]

ALGEBRA PRELIMINARY EXAMINATION
Spring 2012

Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 455 is necessarily cyclic.
2. Let G be a group of order 56. Show that G is solvable.

Part II.

1. Let $f : \mathbb{Q} \to \mathbb{Z}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$. Show that the image of f has at most three elements and there exist an infinite number of such functions whose image has three elements.
2. Let R be a PID and let J denote the intersection of all maximal ideals of R. If $a^2 - a$ is in J for all $a \in R$, then show that R has only finitely many maximal ideals.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be an integral domain and let M, N be projective R-modules. Show that $M \otimes_R N$ is a projective R-module.
2. Suppose R is a principal ideal domain that is not a field. Suppose M is a finitely generated R-module such that for every maximal ideal P of R, M/PM is a cyclic R/P-module. Show that M itself is cyclic.

Part IV.

1. Let $f(X)$ be a monic polynomial of degree 9 having rational coefficients. Assume that $f(X)$ is irreducible in $\mathbb{Q}[X]$. Let K denote the splitting field of f over $\mathbb{Q}$ and let $u \in K$ be a root of f. If $[K : \mathbb{Q}] = 27$, then show that $\mathbb{Q}[u]$ has a subfield L with $[L : \mathbb{Q}] = 3$.
2. Let F, K be fields such that K is a finite Galois extension of F with Galois group G. Suppose $a \in K$ is such that $\sigma(a) - a \in F$ for all $\sigma \in G$. If the characteristic of F does not divide the order of G, then show that $a \in F$. Assuming F to be the field of two elements, construct a quadratic field extension $K := F[a]$ of F such that $\sigma(a) - a \in F$ for all $\sigma \in G$.

Algebra Preliminary Exam **January 2011**

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let G be a finite simple group. Show that if G has a subgroup H with $[G:H] = n \geq 2$, then $|H| \mid (n-1)!$.

2. List, up to isomorphism, all groups of order 153. Justify your answer.

II. 1. Let R be a commutative ring and I an ideal of R. Let $I^* = (I, X)$ be an ideal of the polynomial ring $R[X]$. Determine, in terms of I, when I^* is a prime ideal of $R[X]$ and when I^* is a maximal ideal of $R[X]$. Justify your answers.

2. (a) Show that if a commutative ring R satisfies DCC on ideals (i.e., R is Artinian), then R has only a finite number of maximal ideals.

(b) Give an example to show that (a) may be false if DCC is replaced by ACC (i.e., if R is Noetherian).

III. 1. Let $f: M \to M$ be an R-module homomorphism with $f \circ f = f$. Show that the following statements are equivalent.

(a) f is injective.

(b) f is surjective.

(c) $f = 1_M$.

2. (a) Let G and H be finitely generated abelian groups such that $\mathbb{Z}_n \otimes G \cong \mathbb{Z}_n \otimes H$ for every integer $n \geq 2$. Show that $G \cong H$.

(b) Give an example to show that (a) may be false if G and H are not both finitely generated.

IV. 1. Let F be a subfield of a field L. Show that L/F is an algebraic extension if and only if every subring R of L containing F is a field.

2. Compute the Galois group of $f(X) = X^4 + X + 1 \in \mathbb{Z}_2[X]$.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2011

Attempt all four parts. Justify your answers.

Part I.

1. How many Sylow 2-subgroups does S_5 (the group of permutations of $\{1, 2, 3, 4, 5\}$) have ?

2. Let G be a group of order 231. Show that G is Abelian if and only if G has an Abelian subgroup of order 21.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$, *subrings are assumed to contain* 1 *and ring-homomorphisms are assumed to map* 1 *to* 1.

1. Let R be a UFD such that each maximal ideal of R is a principal ideal. Prove that R is a PID.

2. Let $\mathbb{R}[[X]]$ denote the power-series ring in a single indeterminate X over the field of real numbers $\mathbb{R}$. If T is a multiplicative subset of $\mathbb{R}[[X]]$ containing 1 but not containing 0, then show that either $T^{-1}\mathbb{R}[[X]] = \mathbb{R}[[X]]$ or $T^{-1}\mathbb{R}[[X]]$ is a field.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be an integral domain and I an ideal of R. Show that there exists a surjective R-module homomorphism $f : I \to R$ if and only if I is a nonzero principal ideal.

2. Let K be a field, X an indeterminate over K and M a finitely generated $K[X]$-module. Show that M is a projective $K[X]$-module if and only if M is $K[X]$-module isomorphic to $K[X] \otimes_K V$ for some finite dimensional K-vector space V.

Part IV.

1. Let K be a field and F a subfield of K. The group of units of K is denoted by $K^\times$. Suppose $f \in F[X]$ is a monic irreducible polynomial and $a, b \in K^\times$ are such that $f(a) = 0 = f(b)$. Show that the subgroup of $K^\times$ generated by a, is isomorphic to the subgroup of $K^\times$ generated by b.

2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 4 such that the Galois group of f (over $\mathbb{Q}$) is a group of order 6. Show that f has a root in $\mathbb{Q}$.

Algebra Preliminary Exam **August 2010**

Attempt all problems and justify all answers. All rings have an identity $1 \neq 0$, ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let $f : G \to H$ be a surjective homomorphism of finite groups and $y \in H$ with $|y| = n$. Show that there is an $x \in G$ with $|x| = n$.

2. Let p and q be primes, $p \geq q$, $n \geq 1$, and G a group with $|G| = p^n q$. Show that G has a normal subgroup H of order p^n. (Hint: do the $p > q$ and $p = q$ cases separately.)

II. 1. Let R be a commutative ring with distinct prime ideals P and Q with $P \cap Q = \{0\}$. Show that R is isomorphic to a subring of the direct product of two fields.

2. Let p and q be positive primes. Show that the polynomial $f(X) = X^3 + pX^2 + q \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Q}[X]$.

III. 1. Let A and B be finite abelian groups with $|A| = m$ and $|B| = n$. Show that $\mathrm{Hom}_{\mathbb{Z}}(A,B) = 0$ if and only if $\gcd(m,n) = 1$.

2. Let A be a submodule of a projective R-module B. Show that A is projective if B/A is projective.

IV. 1. Let $K \subseteq F$ and $K \subseteq L$ be subfields of a field M with $[F:K] = p$ and $[L:K] = q$ for distinct primes p and q. Show that $F \cap L = K$, and that $F = K(\alpha)$ and $L = K(\beta)$ for any $\alpha \in F - K$ and $\beta \in L - K$.

2. Let K be a field and $f(X) \in K[X]$ be irreducible and separable with $\deg(f(X)) = n$. Show that if the Galois group G of $f(X)$ over K is abelian, then $|G| = n$.