Analysis Diagnostic Exam May 15, 2019

NAME: ____________________

#1.) ____/15 #2.) ____/20 #3.) ____/15 #4.) ____/15 #5.) ____/15 #6.) ____/20

Total: ____/100

Instructions: There are 100 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (15 points) Let A and B be sets. Prove that

$$A \cap B = A \setminus (A \setminus B),$$

where in general, $C \setminus D = \{x \in C : x \notin D\}$.

2.) For a set $A \subseteq \mathbb{R}$, define $2A = \{2a : a \in A\}$.

(a) (10 points) Suppose $E \subseteq \mathbb{R}$ is closed and non-empty. Prove that $2E$ is closed.

(b) (10 points) Suppose $K \subseteq \mathbb{R}$ is compact and non-empty. Prove that $2K$ is compact.

3.) (15 points) Prove directly from the definition of the limit of a function (that is, without using any limit theorems) that

$$\lim_{x \to 3} x^3 = 27.$$

4.) (15 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable. Suppose f has a local minimum at $c \in (a, b)$. Prove that $f'(c) = 0$. (Do not quote the extreme value theorem; this problem is asking you to prove this version of it.)

5.) (15 points) Suppose $K \subseteq \mathbb{R}$ is compact, $f : K \to \mathbb{R}$ is continuous, and (g_n) is a sequence of functions with $g_n : K \to \mathbb{R}$ for each $n \in \mathbb{N}$. Suppose $g_n \to 0$ uniformly on K. Prove that $fg_n \to 0$ uniformly on K.

6.) Suppose $f : [0, 1] \to \mathbb{R}$ is Riemann integrable (in particular, f is bounded). Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \int_0^x f(t)\, dt.$$

(i) (10 points) Give an example of an f as stated, such that g is not differentiable at $x = 1/2$. You do not have to prove that your function f is Riemann integrable on $[0, 1]$, as long as it is. You do have to prove that the resulting function g is not differentiable at $x = 1/2$.

(ii) (10 points) Prove that there exists an $M \in [0, \infty)$ such that $|g(b) - g(a)| \leq M|b - a|$, for all $a, b \in [0, 1]$. Here M depends on f but not on a or b.