Analysis Diagnostic Exam   May 7, 2021

NAME: ________________________________

#1.) ____/10 #2.) ____/10 #3.) ____/12 #4.) ____/8 #5.) ____/10 #6.) ____/10

#7.) ____/10 #8.) ____/10 Total: ____/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (10 points) Prove directly by using the definition of limits of sequences that

$$\lim_{n\to\infty} \frac{2n^2+3n+1}{n^2+2021} = 2.$$

2.) (10 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and are finite for some $a \in \mathbb{R}$. Use the $\epsilon - \delta$ definition to show that $\lim_{x\to a}[f(x)g(x)]$ exists and

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x).$$

3.) (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \infty.$$

Prove that the set $\{f(x) : x \in \mathbb{R}\}$ is bounded below, and prove that there exists $a \in \mathbb{R}$ such that

$$f(a) = \inf\{f(x) : x \in \mathbb{R}\}.$$

4.) (8 points) For a given number $\alpha > 1$, let

$$f(x) = \begin{cases} |x|^\alpha \sin(1/x) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Prove that f is differentiable at $x = 0$.

5.) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ such that f'' exists and is continuous on $\mathbb{R}$. Assume that $f'(x_0) = 0$ and $f''(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Prove that there is $\delta > 0$ such that

$$f(x) > f(x_0), \quad \forall\, x \in (x_0 - \delta, x_0 + \delta).$$

6.) (10 points) Let

$$f_n(x) = \frac{nx}{1+nx}, \quad \text{for} \quad x \in (0, \infty) \quad \text{and} \quad n \in \mathbb{N}.$$

Prove that $f_n \to 1$ uniformly on $[1, \infty)$ but not uniformly on $(0, \infty)$.

7.) (10 points) Let $f : [0, 2] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} h(x) & x \in [0, 1], \\ g(x) & x \in (1, 2], \end{cases}$$

where $h : [0, 1] \to \mathbb{R}$ and $g : [1, 2] \to \mathbb{R}$ are given bounded, integrable functions. Prove that f is integrable by showing that for every $\epsilon > 0$, there is a partition P of $[0, 2]$ such that

$$U(f, P) - L(f, P) < \epsilon,$$

where $U(f, P), L(f, P)$ are the corresponding upper and lower Darboux sums of f with respect to the partition P.

8.) (10 points) Let A, B be two open sets in $\mathbb{R}$. Only use the definition of open set to prove that $A \cap B$ is open (recall that a set $E \subset \mathbb{R}$ is open if for every $x \in E$, there is $\delta > 0$ such that $(x - \delta, x + \delta) \subset E$).