Analysis Diagnostic Exam May 25, 2022

NAME: ______________________________

#1.)	___/10	#2.)	___/10	#3.)	___/10	#4.)	___/10	#5.)	___/10	#6.)	___/10
#7.)	___/10	#8.)	___/10	Total:	___/80						

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Let $g = f \circ f \circ f$; that is, $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(f(f(x)))$.

(a) (10 points) If g is $1-1$, prove that f is $1-1$.

(b) (10 points) If g is onto, prove that f is onto.

2.) (a) (5 points) Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded below. Let $y \in \mathbb{R}$. Let

$$A + y = \{a + y : a \in A\}.$$

Prove that $A + y$ is bounded below and that $\inf(A + y) = y + \inf A$.

(b) (5 points) Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $y \in \mathbb{R}$. Consider the sequence

$$(a_n + y)_{n=1}^{\infty} = (a_1 + y, a_2 + y, a_3 + y, \cdots).$$

Prove that $(a_n + y)_{n=1}^{\infty}$ is a bounded sequence and $\liminf\,(a_n + y) = y + \liminf\, a_n$.

3.) Prove directly by using the definition of limits of sequences (without using any theorems about limits) that

$$\lim_{n\to\infty} \frac{9n^3 + 2n}{3n^3 + 1,000} = 3.$$

4.) Suppose $A, B \subseteq \mathbb{R}$ with $A \cap B = \emptyset$. Suppose c is a limit point of A and of B. Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are functions. Suppose $\lim_{x\to c} f(x)$ exists, $\lim_{x\to c} g(x)$ exists, and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$. Define $h : A \cup B \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A, \\ g(x), & \text{if } x \in B. \end{cases}$$

Prove that $\lim_{x\to c} h(x)$ exists.

5.) Let A be a subset of $\mathbb{R}$. Prove that

$$\left(\overline{A}\right)^c = (A^c)^\circ .$$

Here $\left(\overline{A}\right)^c$ is the complement of the closure of A and $(A^c)^\circ$ is the interior of the complement of A.

6.) Suppose $E \subseteq \mathbb{R}$ is closed. Suppose $f : E \to \mathbb{R}$ is continuous and $F \subseteq \mathbb{R}$ is closed. Show that

$$f^{-1}(F) = \{x \in E : f(x) \in F\}$$

is closed.

7.) Suppose $a, b \in \mathbb{R}, a < b$, and $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b), with

$$f'(x) = \frac{4f(x)}{1 + (f(x))^2},$$

for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b).

8.) Let $a, b \in \mathbb{R}$ and $a < b$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } a < x < b \\ 1, & \text{if } x = b. \end{cases}$$

(a) (5 points) Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Calculate $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ (the upper and lower Riemann sums for this partition $\mathcal{P}$). Your answer should be expressed in terms of the numbers $x_0, x_1, \ldots, x_{n-1}, x_n$.

(b) (5 points) Is f Riemann integrable on $[a, b]$? Prove your answer.