Analysis Diagnostic Exam May 25, 2023

NAME: ______________________________

#1.) ____/10 #2.) ____/10 #3.) ____/10 #4.) ____/10 #5.) ____/10 #6.) ____/10

#7.) ____/10 #8.) ____/10 Total: ____/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (10 points) Suppose A, B, and C are sets. Prove that

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

2.) (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing, which means that if $x, y \in \mathbb{R}$ and $x \leq y$, then $f(x) \leq f(y)$. Suppose also that f is onto. If A is a non-empty subset of $\mathbb{R}$ and A is bounded above, prove that $f(A)$ is bounded above and

$$\sup f(A) = f(\sup A).$$

3.) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and suppose $A \subseteq \mathbb{R}$.

(a) (5 points) Prove that $f(\,\overline{A}\,) \subseteq \overline{f(A)}$. (For any set $B \subseteq \mathbb{R}$, $\overline{B}$ denotes the closure of B).

(b) (5 points) If A is bounded, prove that $\overline{f(A)} \subseteq f(\,\overline{A}\,)$.

4.) (10 points) Using only the $\epsilon - \delta$ definition of the limit (not using any theorems about the limit or continuity), show that

$$\lim_{x \to 6} \frac{x^2 - 24}{x - 3} = 4.$$

5.) Let A be a subset of $\mathbb{R}$. Suppose that $x_0 \in A$ and x_0 is an isolated point of A. (Recall that a point $x_0 \in A$ is an isolated point of A if there exists $r > 0$ such that $B(x_0, r) \cap A = \{x_0\}$.)

(a) (6 points) If A is closed, prove that $A \setminus \{x_0\}$ is closed.

(b) (4 points) If A is compact, prove that $A \setminus \{x_0\}$ is compact.

6.) (10 points) Suppose A is a subset of $\mathbb{R}$, and $f : A \to \mathbb{R}$ is uniformly continuous on A. If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in A, prove that $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.

7.) (10 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Determine (with proof) whether f is differentiable at $x = 0$.

8.) Define $f : [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & 0 \le x \le 1, \\ 1, & 1 < x \le 2. \end{cases}$$

For $n \in \mathbb{N}$, define the partition P_n of $[0,2]$ by

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{2n-1}{n}, 2\right\} = \left\{\frac{k}{n} : 0 \le k \le 2n\right\}.$$

(a) (7 points) Determine $L(f, P_n)$ and $U(f, P_n)$ (the lower and upper Riemann sums for f with respect to P_n, respectively) for each $n \in \mathbb{N}$.

(b) (3 points) Show that f is Riemann integrable on $[0,1]$ and determine $\int_0^2 f(x)\,dx$.