Analysis Diagnostic Exam August 12, 2022

NAME: ______________________

#1.) ____/10 #2.) ____/10 #3.) ____/10 #4.) ____/10 #5.) ____/10 #6.) ____/10

#7.) ____/10 #8.) ____/10 Total: ____/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (10 points) Suppose X, Y and Z are sets, $f : X \to Y$ and $g : Y \to Z$ are functions, and C is a subset of Z. Prove that

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)).$$

Here we do not assume that f or g is $1-1$ or onto. In general, for $h : U \to V$ and $A \subseteq V$, $h^{-1}(A) = \{u \in U : h(u) \in A\}$ is the inverse image of A under h.

2.) (10 points) Suppose A is a non-empty subset of $\mathbb{R}$, and $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are bounded functions. Prove that $f + g : A \to \mathbb{R}$ is bounded above and

$$\sup_{x\in A} (f + g)(x) \leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

Here $(f + g)(x)$ is defined to be $f(x) + g(x)$ for all $x \in A$. Also $\sup_{x\in A} f(x) = \sup\{f(x) : x \in A\}$ and similarly for $\sup_{x\in A} g(x)$ and $\sup_{x\in A}(f + g)(x)$.

3.) (10 points) Using only the $\epsilon - \delta$ definition of the limit (not using any theorems about the limit or continuity), show that

$$\lim_{x \to 3} \frac{x^2 - 5}{x - 1} = 2.$$

4.) (10 points) Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers, and $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$. Prove that

$$\limsup x_{n_k} \leq \limsup x_n.$$

5.) (a) (6 points) Suppose $E \subseteq \mathbb{R}$ is non-empty and bounded above. Prove that $\sup E \in \overline{E}$. Here $\overline{E}$ is the closure of E.

(b) (4 points) Suppose $K \subseteq \mathbb{R}$ is non-empty and compact. Prove that $\sup K \in K$.

6.) (10 points) Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Suppose that for each $n \in \mathbb{N}$, there exists $x_n \in [-1,1]$ such that

$$f(x_n) = x_n^3 + \frac{x_n}{n} + 2.$$

Prove that there exists $x \in [-1,1]$ such that $f(x) = x^3 + 2$.

7.) (10 points) Suppose $a, b \in \mathbb{R}$ with $a < b$ and $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) with

$$f'(x) = \frac{1}{1 + (f(x))^4},$$

for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b).

8.) (10 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Recall that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if: for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Here $U(f, P)$ and $L(f, P)$ are the upper and lower sums, respectively, of f with respect to P.

Prove that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, and $c > 0$ is a constant, then cf is Riemann integrable on $[a, b]$. You can assume elementary properties of suprema and infima for this problem.