Analysis Preliminary Exam – August 2023

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Prove or disprove: There is a holomorphic function f on
$$P = \{z \in \mathbb{C} : 0 < |z| < 1\}$$
such that f' has a simple pole at $z = 0$.

2. Let f_n be Lebesgue measurable functions on $[0,1]$ such that $\|f_n\|_{2023} \leq 2023$ for all $n \geq 1$ and f_n converges to 0 pointwise almost everywhere on $[0,1]$. Prove that $\int_0^1 |f_n| dm \to 0$.

3. Let $\Omega = \{x + iy \,:\, x > 0 \text{ and } y > 0\} \setminus \{re^{i\pi/4} \,:\, r \geq 1\}$. Determine an explicit conformal map from Ω onto $\mathbb{D}$. You may express your answer as a composition of explicit maps.

4. Let $f \in L^1(\mathbb{R})$ and $p > 0$. Prove that
$$\lim_{n\to\infty} n^{-p} f(nx) = 0$$
for m-a.e. $x \in \mathbb{R}$.

   Hint: Show that $\sum_{n\in\mathbb{N}} n^{-p} f(nx)$ converges a.e..

5. Consider a rational function $f = p/q$ where q is a polynomial of degree $n \geq 2$ and p is a polynomial of degree $n - 2$ or less. Let $z_1, \ldots, z_m$ be the distinct zeros of q. Show that $\sum_{k=1}^{m} \operatorname{Res}(f, z_k) = 0$.

6. Let $p > 1$ and $f : [0,1] \to \mathbb{R}$ be a function such that
$$\sum_{i=1}^{n} \frac{|f(b_i) - f(a_i)|^p}{(b_i - a_i)^{p-1}} \leq 2023$$
whenever $(a_1, b_1), \ldots, (a_n, b_n)$ disjoint intervals in $[0,1]$. Prove that f is absolutely continuous.

7. Let $g : \mathbb{D} \to \{z \,:\, |z| \leq 5\}$ be holomorphic with $g(0) = 2i$. Prove that g has no zeros in the set $\{z \,:\, |z| \leq 1/5\}$.

8. Let $(f_n)_{n\in\mathbb{N}}$ be Lebesgue measurable nonnegative functions on $[0,1]$. Show that there exist constants $c_n > 0$ such that $\sum_{n\geq 1} c_n f_n(x)$ converges for almost all $x \in [0,1]$.

9. Let $f_n : \mathbb{D} \to \mathbb{D}$ be analytic functions such that $f_n \to 0$ pointwise on $\{z \in \mathbb{C} : |z| < 1/2\}$. Show that $f_n \to 0$ locally uniformly in $\mathbb{D}$ (in other words, show that f_n converges to 0 uniformly on each compact subset of $\mathbb{D}$).

Analysis Prelim, January 2023

The exam has 9 problems. In the problems below m is used to denote Lebesgue measure on $\mathbb{R}$. If $a \in \mathbb{C}$ and $R > 0$, then $B(a, R) = \{z \in \mathbb{C} : |z - a| < R\}$.

1. For each integer $n \geq 1$ denote by b_n the Lebesgue measure of the unit ball centered at the origin in $\mathbb{R}^n$.

(i) Show that $b_{n+1} = b_n \cdot \int_{-1}^{1} (\sqrt{1-t^2})^n dt$ for all $n \geq 1$.

(ii) Show that $b_n \to 0$ as $n \to \infty$.

2. Let f be analytic and non-constant in the disk $B(0, 2)$. Suppose that for all z with $|z| = 1$, we have $|f(z)| = 1$. Show that f has at least one zero in $B(0, 1)$.

3. Let $(f_n) \subseteq L^2[0, 1]$ be a sequence of functions such that

$$\lim_{n,k\to\infty} \int_{[0,1]} |f_n - f_k|^2 \, dm = 0.$$

Let also $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. For each $n \in \mathbb{N}$ and $x \in [0, 1]$, define

$$g_n(x) = \int_{[0,1]} K(x, y) f_n(y) \, dm(y).$$

Prove that the sequence (g_n) converges uniformly on $[0, 1]$.

4. Let f and g be meromorphic functions in $\mathbb{C}$. Assume that

$$|f(z) + g(z)| \leq |g(z)|$$

for every $z \in \mathbb{C}$ which is not a pole of either f or g. Show that there is a constant c with $|c + 1| \leq |c|$ such that $f(z) = cg(z)$.

5. Suppose that $f, g \in L^1(\mathbb{R}, m)$ and let $g_n(x) = g(x - n)$ for all $n \geq 1$ and all $x \in \mathbb{R}$. Show that $\lim_{n\to\infty} \|f + g_n\|_1 = \|f\|_1 + \|g\|_1$.

6. Suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in the unit disk $B(0, 1)$. Suppose also that f extends to be meromorphic in $B(0, R)$ for some $R > 1$ with finitely many poles, all of which lie on the unit circle and are simple poles. Prove that the sequence (a_n) is bounded.

7. Let (X, μ) be a finite measure space. Let $(f_n)_{n\geq 1}$ be a sequence of integrable functions on X. Suppose that there is an integrable function f on X such that $(f_n(x))$ converges to $f(x)$ pointwise almost everywhere on X.

Prove that, for every $\varepsilon > 0$, there are $M > 0$ and a measurable subset E of X such that $\mu(E) < \varepsilon$ and $|f_n(x)| < M$ for all $x \in X \setminus E$ and all $n \geq 1$.

8. Let $G \subsetneq \mathbb{C}$ be a simply connected open set with $0 \in G$, and $f : G \to G$ be analytic with $f(0) = 0, f'(0) = 1$. Show that $f(z) = z$.

Does the same conclusion hold for $G = \mathbb{C}$? (Prove your answer.)

9. Let μ be a positive measure on the measurable space $(X, \mathcal{M})$, and let $f : X \to X$ be a measurable transformation, i.e. $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{M}$. For $E \in \mathcal{M}$ define $\mu_f(E) = \mu(f^{-1}(E))$.

(a) Show that μ_f is a measure on $(X, \mathcal{M})$.

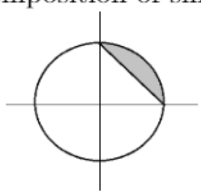
(b) Prove that $\int_X h(f(x))d\mu(x) = \int h(y)d\mu_f(y)$ for every $h \in L^1(\mu_f)$.

Analysis Preliminary Exam, August 2022

1. Let f be an entire function and suppose that for each $z_0 \in \mathbb{C}$ there is an integer n such that $a_n = 0$ in the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Show that f must be a polynomial.

2. Let $A \subset [0,1]$ be a Lebesgue measurable set of positive measure. Show that there exist $x \neq y$ in A such that $x - y$ is rational.

3. Let $G = \{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Re} z + \mathrm{Im} z > 1\}$. Determine a conformal mapping f from G onto the open unit disc $\mathbb{D}$. You may express f as a composition of simpler maps.



4. Let $(r_n)_{n\geq 1}$ be an enumeration of all the rationals in $[0,1]$. Show that the function

$$f(x) = \sum_{n\geq 1} \frac{1}{n^2 \sqrt{|r_n - x|}}$$

is finite almost everywhere with respect to the Lebesgue measure (for x real).

5. Let $R > 1$ and $f : B(0,R) \to \mathbb{C}$ be analytic. Show $\partial f(B(0,1)) \subseteq f(\partial B(0,1))$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f_n $(n \geq 1)$ and f be measurable, real-valued functions on X. We say that f_n converges almost uniformly to f if for every $\varepsilon_1, \varepsilon_2 > 0$ there is a set E and a positive integer N such that $\mu(E) < \varepsilon_1$ and $|f_n(x) - f(x)| < \varepsilon_2$ for all $n \geq N$ and all $x \in E^c$.

Show that if f_n converges almost uniformly to f then f_n converges pointwise a.e. to f and f_n converges in measure to f.

7. Let $E \subset \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ be a Lebesgue measurable set such that $m(E_x) \geq x^3$ for all $0 \leq x \leq 1$.

(i). Show that there is $y \in [0,1]$ such that $m(E^y) \geq \frac{1}{4}$.

(ii). Prove a stronger inequality than (i), by finding a constant $c > \frac{1}{4}$ such that for every set E satisfying the hypothesis there is $y \in [0,1]$ with $m(E^y) \geq c$.

8. Show that there is a holomorphic function $f(z)$ on a neighborhood of 0 such that $f(z)^2 = \frac{\sin(z)}{z}$ and determine the radius of convergence of the power series of $f(z)$ at 0 (with proof).

9. Find all $q \geq 1$ such that $f(x^2) \in L^q((0,1), m)$ for all $f(x) \in L^4((0,1), m)$.

ANALYSIS PRELIMINARY EXAM – JANUARY 2022

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

1. Show that for any $\epsilon \in [0, 1)$ there is a constant $C < \infty$ depending only on ϵ such that if $f : \mathbb{D} \to \mathbb{C}$ is analytic, then for all $z \in \mathbb{D}$ with $|z| \leq \epsilon$ we have

$$|f'(z)| \leq C \int_{\mathbb{D}} |f(x + iy)| \, dy \, dx.$$

2. Let $G \subset \mathbb{C}$ be an open simply connected domain that is not $\mathbb{C}$, and let $f : G \to G$ be analytic but not the identity. Show that f has at most one fixed point (that is, there exists at most one $z \in G$ such that $f(z) = z$).

3. Let g be a real-valued measurable function on $[0, 1]$. Assume that for any $f \in L^1([0, 1])$ we have $fg \in L^1([0, 1])$. Show that $g \in L^\infty([0, 1])$.

4. Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. How many solutions does the equation

$$a - z - e^{-z} = 0$$

have on the half-plane $\{z : \operatorname{Re} z > 0\}$?

5. Find with proof the limit

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n}{1 + x} \, dx.$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let E be the set of all points $x \in \mathbb{R}$ such that f is continuous at x. Show that E is a Borel set.

7. Prove that there is no one-to-one analytic function mapping the annulus $\{z : 0 < |z| < 1\}$ onto the annulus $\{z : 1/2 < |z| < 2\}$.

8. Let $E \subset \mathbb{R}$ be a nonempty Borel measurable set and let $f \in L^1(E)$. Show that for each $0 \leq a \leq \int_E |f| \, dm$, there exists a nonempty Borel measurable set $E_a \subset E$ such that $\int_{E_a} |f| \, dm = a$.

9. Does there exist an entire function f such that $f(0) = 0$, $f(i) = i$, and $|f(z)| \leq |z|^{2/3}$ for all $z \in \mathbb{C}$? Justify your answer.

ANALYSIS PRELIMINARY EXAM – AUGUST 2021

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

1. Construct a 1-to-1 conformal map of the upper half-plane $\mathbb{H}$ onto the domain

$$D = \left\{ z \in \mathbb{C} : |z| > 1 \text{ and } |z - i| < \sqrt{2} \right\}.$$

A sequence of explicit functions and the order in which they are to be composed to give the final mapping will suffice.

2. Given $a \in \mathbb{R}$, compute (with proof) the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2}\, dx.$$

3. Let $K(x, y)$ be Lebesgue measurable on $\mathbb{R}^2$ such that for some $C > 0$

$$\int_{-\infty}^{\infty} |K(x, y)|\, dy \leq C, \qquad \text{for a.e. } x \in \mathbb{R},$$

$$\int_{-\infty}^{\infty} |K(x, y)|\, dx \leq C, \qquad \text{for a.e. } y \in \mathbb{R}.$$

For $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, define

$$Tf(x) = \int_{-\infty}^{\infty} K(x, y) f(y)\, dy.$$

Show that $Tf \in L^p(\mathbb{R})$ and that $\|Tf\|_p \leq C\|f\|_p$.

4. Let μ be a finite measure on a measurable space $(X, \mathcal{M})$ and suppose that $\{E_n\}_{n\in\mathbb{N}}$ are measurable sets with $\mu(E_n) \geq \alpha$ for all $n \in \mathbb{N}$. Let $E = \{x \in X : \ x \in E_n \text{ for infinitely many } n\}$. Show that E is measurable and that $\mu(E) \geq \alpha$.

5. Let

$$f(z) = \frac{1}{z - 1} - \frac{1}{z + 1} = \frac{2}{z^2 - 1}$$

defined on the domain $U = \mathbb{C} \setminus [-1, 1]$. Show that $\int_\gamma f(z)\, dz = 0$ for any closed rectifiable curve γ in U.

6. Let U be a connected open subset of $\mathbb{C}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions defined on U. Suppose that $f_n \to f$ uniformly on compact subsets of U and that the functions f_n are nonvanishing on U. Show that, either f is nonvanishing, or f is identically zero.

7. Let $f : [0, 1] \times \mathbb{D} \to \mathbb{D}$ be a measurable function such that $z \mapsto f(t, z)$ is holomoprhic for all $t \in [0, 1]$. Show that the function

$$F(z) = \int_0^1 f(t, z)\, dt$$

is holomorphic in $\mathbb{D}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function such that

$$\sup_{n\in\mathbb{N}} \int_{-\infty}^{\infty} |x|^n f(x)\, dx < \infty.$$

Show that $f(x) = 0$ for a.e. $x \in (-\infty, -1) \cup (1, \infty)$.

9. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{R})$ such that for all continuous and compactly supported functions g

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\, dx = g(0).$$

Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ is not Cauchy in $L^1(\mathbb{R})$.

PRELIMINARY EXAMINATION IN ANALYSIS–AUGUST 2020

Notation: $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.

1. Construct a 1-1 conformal map of $\mathbb{D} \cap \{Re(z) > 0\}$ onto $\mathbb{D}$.

2. If $f \in H(\mathbb{D})$ wiith $f(\mathbb{D}) \subset \mathbb{D}$, how big can $|f'(\frac{1}{2})|$ be? (You should explicitly display and extremizing function.)

3. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx.$$

4. Find, with proof:

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n^2 x}{1+x^2} e^{-n^2x^2} dx.$$

5. let $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Show that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $fg \in L^r(\mathbb{R})$ and:

$$||fg||_{L^r} \leq ||f||_{L^p} ||g||_{L^q}.$$

6. Suppose that $||f_n||_{L^2[0,1]} \leq 1$ for $n = 1, 2, \ldots$ and $f_n \to 0$ a.e. Show that $||f_n||_{L^1[0,1]} \to 0$. (*Hint:* use Egorov's theorem.)

7. Find a closed set $C \subset L^2([0,1])$ with $\inf_{f \in C} ||f||_{L^2[0,1]} = 1$ but $||f||_{L^2[0,1]} > 1$, for any $f \in C$.

8. Show that $\forall \epsilon > 0 \exists \delta > 0$ with the following property:
If $f \in H(\mathbb{D})$ with $f(\mathbb{D}) \subset \mathbb{D}$ and $|f(x)| < \delta$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then $|f(\frac{1}{2}i)| < \epsilon$.
Hint: use a normal family argument.

9. Let $f \in L^2[-1,1]$. Show that $\forall z \in \mathbb{C}$ the function $t \mapsto f(t)e^{itz}$ is integrable, and that:

$$F(z) = \int_{-1}^{1} f(t) e^{itz} dt$$

is an entire function.

Analysis Preliminary Exam – January 2019

1. Evaluate the following (with proof): $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$.

2. Let $g \in L^2(0,\infty)$, and let $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{ Re } z > 0\}$.

   (a) Show that for each $z \in \mathbb{C}_+$ the function $\frac{g(t)}{1+zt}$ is in $L^1(0,\infty)$.

   (b) Define $f : \mathbb{C}_+ \to \mathbb{C}$ by $f(z) = \displaystyle\int_0^\infty \frac{g(t)}{1+zt}dt$ and show that f is continuous on $\mathbb{C}_+$.

3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : \mathbb{D} \to \mathbb{C}$ be analytic. For $0 \le r < 1$ define
$$A(r) = \max_{|z|=r} \text{ Re } f(z).$$
   Show that $A(r)$ is strictly increasing unless f is constant.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, with $\mu(X) = \infty$. Show that $(X, \mathcal{M}, \mu)$ is σ–finite if and only if there exists a measurable function $f : X \to (0, \infty)$ such that $\int_X f d\mu = 1$.

5. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Suppose that $f : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ satisfies the following:

   (i) f is continuous.

   (ii) f is holomorphic on $\mathbb{H}$.

   (iii) $f(z)$ is real whenever z is real.

   (iv) $f(\mathbb{H}) \subseteq \mathbb{H}$.

   Show that $f(\mathbb{H})$ is a dense subset of $\mathbb{H}$.

6. Let m be Lebesgue measure, and set $\mathcal{X} = \{f : [0,1] \to \mathbb{R},\ f \text{ Lebesgue measurable}\}$, where functions that are equal m-a.e. are identified. Define the distance d on $\mathcal{X}$ by $d(f,g) = \int_0^1 \frac{|f-g|}{|f-g|+1}dm$. It is known that $(\mathcal{X}, d)$ is a metric space. Let $f_n, f \in \mathcal{X}$.

   (a) Show that if $f_n \to f$ pointwise a.e., then $f_n \to f$ in the topology given by the distance d.

   (b) Show that the converse of (a) is false, i.e. convergence in d does not imply pointwise a.e. convergence.

7. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{ Re } z > 0\}$ and let $w \in \mathbb{C}_+$. If $f : \mathbb{C}_+ \to \mathbb{C}_+$ is analytic, then show
$$|f'(w)| \le \frac{\text{Re } f(w)}{\text{Re } w}.$$

8. Let $(X, \mathcal{A})$ be a measurable space, and let μ and ρ be positive, finite measures on $(X, \mathcal{A})$. Suppose that $\mu \ll \rho$. Prove that $\mu \times \mu \ll \rho \times \rho$ and
$$\frac{d(\mu\times\mu)}{d(\rho\times\rho)}(x,y) = \frac{d\mu}{d\rho}(x)\cdot\frac{d\mu}{d\rho}(y)$$
   where we follow the convention that functions that are equal a.e. are identified.

9. Let $G = \{z \in \mathbb{C} : |\text{Im } z| < 2\}$ and let $f : G \to \mathbb{C}$ be a bounded analytic function such that $\lim\limits_{x\to+\infty} f(x) = c$. Show that $\lim\limits_{x\to+\infty} f(z+x) = c$ for all $z \in G$.

Analysis Preliminary Exam – Fall 2018

1. Let G be a non-empty, connected, open subset of $\mathbb{C}$. Fix a point $\alpha \in G$, and let $\{\alpha_n\}$ be a sequence of points in G that converges to α. Let f and g be holomorphic functions on G that do no vanish at any point of G. Show that if
$$\frac{f'(\alpha_n)}{f(\alpha_n)} = \frac{g'(\alpha_n)}{g(\alpha_n)}$$
for every n, then g is a multiple of f.

2. Find (with proof) the following limit:
$$\lim_{n\to\infty} \int_0^\infty \frac{1}{1 + x^{\frac{n}{\ln(n+2018)}}}\, dx.$$

3. Let $\Omega = \{z \in \mathbb{C} : |z-1| > \sqrt{2} \text{ and } |z+1| > \sqrt{2}\}$. Explicitly give a conformal mapping ϕ of Ω onto the punctured unit disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$. (A sequence of explicit functions whose composition gives ϕ will suffice.)

4. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space. If $f \in L^1(m)$ and $g \in L^p(m)$, for some $p \in [1, \infty)$, prove that
$$||f * g||_{L^p} \leq ||f||_{L^1}||g||_{L^p}.$$
(Recall $f * g(x) = \displaystyle\int f(x-y)g(y)\, dy$.)

5. Let M_n be a sequence of positive numbers and let
$$\mathcal{F} = \{f : f \text{ is holomorphic on } \mathbb{D} \text{ and } |f^{(n)}(0)| \leq M_n \text{ for } n = 0, 1, 2, \cdots\}.$$
Show that $\mathcal{F}$ is a normal family if and only if $\displaystyle\sum_{n=0}^\infty \frac{M_n z^n}{n!}$ converges for all $z \in \mathbb{D}$.

6. Let f be a nonnegative Lebesgue measurable function on $[0, \infty)$ such that $\int_0^\infty f(x)\, dx < \infty$. Show there exists a positive, increasing, measurable function ϕ on $[0, \infty)$ with $\displaystyle\lim_{x\to\infty} \phi(x) = \infty$ such that
$$\int_0^\infty \phi(x) f(x)\, dx < \infty.$$

7. Let f be holomorphic in an open set containing the closed unit disc $\{z : |z| \leq 1\}$, with $f(i/5) = 0$ and
$$|f(z)| \leq |e^z| \text{ for all } z \text{ with } |z| = 1.$$
How large can $|f(-i/5)|$ be? Find (with proof) the best possible upper bound.

8. Suppose (X, Σ, μ) is a finite measure space and $\{f_n\}$ and f are (X, Σ, μ) measurable functions such that $f_n \to f$ in measure and $|f_n(x)|, |f(x)| < \infty$ a.e. Prove that $f_n^2 \to f^2$ in measure.

9. Find (with proof) all functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ that have a simple pole at i and at ∞, but are holomorphic elsewhere.

Analysis Prelim August 2017

1. Let (X, Σ, μ) be a measure space, and let f_n, f be measurable functions with $f_n \to f$ a.e., and such that there is an $F \in L^1(\mu)$ such that for each n $|f_n| \le F$ on X. Show that $f_n \to f$ in measure.

Recall that f_n is said to converge to f in measure, if for each $\varepsilon > 0$ there is N such that $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$ for all $n \ge N$.

2. Let $\log z$ be the principal branch of the logarithm. The function

$$f(z) = \frac{z}{(2 + \log z)^2}$$

has one pole at a point $p \in \mathbb{C}$. Determine p, the singular part S of f at p, and the radius of convergence of the power series of $g = f - S$ at p. (You do not have to determine the power series of g.)

3. Find with proof

$$\lim_{n\to\infty} \int_0^\infty \frac{n^2 \sin \frac{x}{n}}{n^3 x + x(1 + x^3)} dx.$$

4. For $\varepsilon > 0$, let $S_\varepsilon = \{x \in \mathbb{R} : |x - 1| > \varepsilon\}$. Use the residue theorem to determine

$$\lim_{\varepsilon\to 0^+} \int_{S_\varepsilon} \frac{x}{(x^2 + 4)(x - 1)} dx.$$

Make sure to justify your work.

5. Let (X, Σ, μ) be a finite measure space, and let f_n, f be measurable functions such that $f_n \to f$ a.e.

Suppose that the sequence $\{f_n\}$ has the following property: For each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever E is a measurable set with $\mu(E) < \delta$, then $\int_E |f_n| d\mu < \varepsilon$ for all n.

Show that $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.

6. Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic for all z except for poles at $z = a_1, a_2, \cdots, a_k$. Assume that f has the Laurent expansion $f(z) = \sum_{j=-\infty}^{n} b_j z^j$ valid for $|z| > \max_i\{|a_i|\}$. Here $n < \infty$. Show that f is a rational function.

7. Let m be Lebesgue measure on $[0, 1]$ and let $E \subseteq [0, 1] \times [0, 1]$ be an $m \times m$- measurable set. Suppose

$$m(\{x : m(E_x) \ge 1/3\}) \ge 1/2.$$

Show that $(m \times m)(E) \ge 1/6$ and give an example of a set E, where equality is attained. Here $E_x = \{y \in [0, 1] : (x, y) \in E\}$.

8. Suppose f is an analytic function mapping the unit disc to itself with $f(0) = 0$. Suppose also that f has "radial limit" $f^*(1)$ at $z = 1$, meaning $f^*(1) = \lim_{x\to 1} f(x)$. We define the "angular derivative" of f at $z = 1$ as $f'(1) = \lim_{x\to 1} \dfrac{f^*(1) - f(x)}{1 - x}$, if this limit exists. It is understood that in both of these limit expressions $x \in \mathbb{R}$, $0 \leq x < 1$.

Prove that if $|f^*(1)| = 1$ and if the angular derivative $f'(1)$ exists, then $|f'(1)| \geq 1$.

9. Let $f \in L^1([0,1], dx)$ and g be a bounded Lebesgue measurable periodic function on $\mathbb{R}$ with period 1, i.e. $g(x) = g(x+1)$ for all $x \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \int_0^1 f(x)dx \cdot \int_0^1 g(x)dx.$$

Hint: Try a continuous function f first.

Analysis Prelim August 2016

For $a \in \mathbb{C}$ and $r > 0$ let $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{N} = \{1, 2, ...\}$.

1. Suppose that $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, $1 \leq p < \infty$ and $f_n \to f$ in L^p. Suppose that $\{g_n\}_{n=1}^{\infty}$ is a sequence of μ-measurable functions that converges point-wise a.e. in X to $g : X \to \mathbb{R}$, with the further property that $|g_n| \leq M < \infty$, for some M and all $n \in \mathbb{N}$.
Prove that $g_n \cdot f_n \to g \cdot f$ in L^p.

2. Prove that $\frac{1}{2\pi} \int_0^{2\pi} e^{\cos t} dt = \sum_{n=0}^{\infty} \frac{1}{(2^n n!)^2}$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that $f : X \to \mathbb{R}$ is non-negative and μ-integrable. Define for every $A \in \mathcal{A}$, $\nu(A) := \int_A f \, d\mu$.

(i) Prove that ν is a measure on $\mathcal{A}$.

(ii) Prove that if $g : X \to \mathbb{R}$ is ν-measurable and ν-integrable, then $f \cdot g$ is a μ-measurable, μ-integrable function and

$$\int g \, d\nu = \int f \cdot g \, d\mu.$$

4. Let $w \in \mathbb{D}$ and $f : \mathbb{D} \to \mathbb{D}$ be analytic and such that $f(0) = f(w) = 0$.
(a) Prove that $|f'(0)| \leq |w|$.
(b) Determine all such f with $|f'(0)| = |w|$.

5. Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous and assume that $f' \in L^2([0, 1])$ and $f(0) = 0$. Show that the following limit exists and compute it:

$$\lim_{x \to 0+} x^{-1/2} f(x).$$

6. Let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions such that $f_n(z) \to 1$ for one $z \in \mathbb{D}$. Prove that $f_n \to 1$ uniformly on each compact subset of $\mathbb{D}$.

7. Let $M > 0$, let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_n \in L^2(\mu)$ with $\int_X |f_n|^2 d\mu \leq M$ for all $n \in \mathbb{N}$.
Show: If $\{a_n\} \in l_2$, then $a_n f_n(x) \to 0$ a.e. $[\mu]$.

8. Let $r > 0$ and $f : B(0, r) \setminus \{0\} \to \mathbb{C}$ be analytic with Re $f(z) > 0$ for all $z \in B(0, r) \setminus \{0\}$.
Show that f has a removable singularity at 0.

9. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx = 0.$$

Analysis Prelim January 2016

For $a \in \mathbb{C}$ and $r > 0$ let $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$. For a region $G \subseteq \mathbb{C}$ Hol(G) denotes the analytic functions on G.

1. Use the Residue theorem to calculate $\int_{-\infty}^{\infty} \frac{\cos 2t}{t^2-2t+2} dt$.

2. With proof determine the limit as $n \to \infty$ of

$$\int_0^\infty \frac{\cos \frac{x^2}{n}}{(1+x^n)\sqrt{x}} dx.$$

3. Does there exist an analytic function mapping the unit disc onto the whole complex plane? If no, explain why not; if yes, describe an example.

4. Show that

$$f \to \int_0^1 \int_0^x f(y)(x+y)^{-5/4} dy dx$$

defines a continuous linear functional on $L^2[0,1]$.

5. Let f be analytic on $B(0,1)$ with Re $f(z) > 0$ for all $z \in B(0,1)$. Show that $|f'(0)| \leq 2$ Re $f(0)$.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.
Show: If $f_n, f, g \in L^1(\mu)$ such that
(a) $|f_n(x)| \leq g(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, and
(b) $f_n \to f$ in measure,
then $f_n \to f$ in $L^1(\mu)$.

7. Let $\Omega = B(0,1) \setminus \{0\}$, and suppose that $f, g \in$ Hol(Ω) with $f = e^g$. Show that if f does not have an essential singularity at 0, then f must have a removable singularity at 0 and $\lim_{z\to 0} f(z) \neq 0$.

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.
(a) For $a \in \mathbb{C}$ and $r \in \mathbb{R}$ define the half space

$$H(a,r) = \{z \in \mathbb{C} : \text{ Re } az \leq r\}.$$

Show: If $f : X \to H(a,r)$ is measurable, then $\frac{1}{\mu(A)} \int_A f d\mu \in H(a,r)$ for all $A \in \mathcal{M}$ with $\mu(A) \neq 0$.
(b) Show: If $f : X \to \mathbb{C}$ is measurable such that $\frac{1}{\mu(A)} \left|\int_A f d\mu\right| \leq 1$ for all $A \in \mathcal{M}$ with $\mu(A) \neq 0$, then $|f(x)| \leq 1$ a.e.

9. Let $G = \{z \in \mathbb{C} : -1 < \text{ Im } z < 1\}$ and let $f \in$ Hol(G) be such that
(a) $|f(z)| \leq \frac{1}{1-|\text{ Im } z|}$ for all $z \in G$, and
(b) $lim_{x\to\infty} f(x) = 0$.
Set $f_n(z) = f(z+n)$ and show that $f_n \to 0$ locally uniformly on G.

Real and Complex Preliminary Exam
August 2015

1. A complex-valued function f on the plane is said to be **locally M-Lipschitz** for $M > 0$ if for each $z \in \mathbb{C}$ there exists an $\epsilon > 0$ so that $|f(w)-f(z)| < M|w-z|$ whenever $|w-z| < \epsilon$. Given $M > 0$, state and prove a description of all entire functions f which are locally M-Lipschitz on $\mathbb{C}$.

2. With proof find the limit as $n \to \infty$ of

$$\int_0^n \frac{x \sin(\frac{1}{nx})}{\sqrt{x^2+1}} dx.$$

3. Let $\Omega = \{z \in \mathbb{C} : z \neq it \text{ for any } t \geq 0\}$, and define g by

$$g(z) = \frac{z}{(z+1)^2}, \qquad z \in \Omega \setminus \{-1\}.$$

(a) Show that there is an analytic branch of $\sqrt{g(z)}$ in $\Omega \setminus \{-1\}$.

(b) Fix a branch f of $\sqrt{g(z)}$ with $f(1) = 1/2$. Determine the nature of the singularity of f at $z = -1$ and calculate the residue of f at -1.

4. Suppose that F is a nonnegative function that is integrable on $\mathbb{R}$ (with respect to Lebesgue measure dm) and that there is a constant C such that

$$\int_{\mathbb{R}} Ff \, dm \leq C \int_{\mathbb{R}} f \, dm$$

whenever f is a nonnegative continuous function on $\mathbb{R}$ having compact support. Prove that $F(x) \leq C$ for almost all x.

5. Let Ω be the open unit disc $\mathbb{D}$ with the real segment $[\frac{1}{2}, 1)$ removed. Construct an explicit conformal mapping f from Ω onto $\mathbb{D}$ with $f(0) = 0$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n, g_n, f, g : X \to \mathbb{C}$ be measurable functions satisfying:

(a) $g_n, g \in L^1$, $g_n \to g$ in $L^1(\mu)$, and $g_n(x) \to g(x)$ a.e.

(b) $|f_n(x)| \leq |g_n(x)|$ for all $x \in X$ and all $n \in \mathbb{N}$,

(c) $f_n(x) \to f(x)$ a.e. $[\mu]$.

Show that $f_n \to f$ in $L^1(\mu)$.

7. Let Ω be a connected open subset of the plane. Suppose $f : \Omega \to \mathbb{C}$ is a continuous complex function having line integrals in Ω which are independent of path. Prove that there exists a function F analytic on Ω such that $F' = f$.

8. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $f : X \to \mathbb{C}$ be measurable. Let m denote Lebesgue measure on $\mathbb{R}$. Show that $\int_X |f|^p \, d\mu = p \int_{(0,\infty)} t^{p-1} \mu(\{|f| > t\}) \, dm(t)$ for all $p > 0$.

9. Let f be an analytic function mapping the unit disc $\mathbb{D}$ to itself with $f(0) = 0$ and $|f'(0)| < 1$. Let $f^{(n)} = f \circ f \circ \ldots \circ f$ be the function obtained by composing f with itself n-times. Prove that $f^{(n)} \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Analysis Preliminary Exam Spring 2015

Instructions: All problems have equal weight. Throughout the exam, m denotes Lebesgue measure on $\mathbb{R}$, and $\mathbb{H}$ denotes the upper halfplane $\{x+iy \in \mathbb{C} : y > 0\}$.

1. For $x \in (0,\infty)$ and $n \in \{1,2,3,\dots\}$, let
$$f_n(x) = \frac{e^{\sin(x^2/n)}}{1+x}.$$
Evaluate, with proof:
(A) $\lim_{n\to\infty} \int_0^n f_n^2 \, dm$
(B) $\lim_{n\to\infty} \int_0^n f_n \, dm$.

2. Suppose $p \in (1,\infty)$, $f \in L^p((0,\infty), dm)$, and $0 < s < 1/p$. Prove that
$$\lim_{N\to\infty} \frac{1}{N} \int_0^N x^s f(x) \, dm(x) = 0.$$

3. Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable. Let $\|f\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]} |f(t)|$. Assume that
$$0 < \|f\|_\infty < \infty.$$
Prove that
$$\lim_{n\to\infty} \frac{\int_{[0,1]} |f|^{n+1} \, dm}{\int_{[0,1]} |f|^n \, dm} = \|f\|_\infty.$$
You may assume the fact that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

4. Let $\mathcal{M}$ denote the Lebesgue measurable subsets of $[0,1]$. Suppose μ is a (positive) measure on $([0,1], \mathcal{M})$ and $\mu(\{0\}) = 0$. Define
$$f(x) = \mu([0,x]),$$
for $0 \le x \le 1$. Suppose that f is absolutely continuous on $[0,1]$.
(A) Prove that $\mu << m$.
(B) Prove that $\int_E f' \, dm = \mu(E)$ for all $E \in \mathcal{M}$.

5. Let m_2 denote Lebesgue measure on $\mathbb{R}^2$. Let $E \subseteq [0,1] \times [0,1]$ be m_2-measurable, and suppose that

$$m(\{y \in [0,1] : (x,y) \in E\}) = 1 \text{ for } m - a.e.\, x \in [0,1].$$

Prove that

$$m(\{x \in [0,1] : (x,y) \in E\}) = 1 \text{ for } m - a.e.\, y \in [0,1].$$

6. Let $\Omega = \{z : |z| < 1 \text{ and } |\mathrm{Im}(z)| > \mathrm{Re}(z)\}$. Give an explicit example of an unbounded harmonic function ϕ on Ω that extends continuously to $\partial\Omega \setminus \{0\}$ and vanishes there. (A sequence of explicit functions whose composition gives ϕ will suffice.)

7. Determine the following integral and justify your answer: $\displaystyle\int_0^\infty \frac{1+x^2}{1+x^4}\, dx.$

8. Let f be an entire function whose range omits the negative real axis. Prove that f is constant.

9. Let G be a simply connected domain that is a proper subset of $\mathbb{C}$, let $z_0 \in G$, and let

$$\mathcal{F} = \{f : \mathbb{H} \to G,\ f \text{ is holomorphic, and } f(i) = z_0\}.$$

Prove that $\mathcal{F}$ is normal.

Analysis Preliminary Exam Fall 2014

Instructions: All problems have equal weight. Throughout the exam, dm denotes Lebesgue measure on $\mathbb{R}$.

1. Find the exact value (with proof) of

$$\lim_{n\to\infty}\int_0^\infty \frac{x^n}{x^{n+3}+1}\,dm(x).$$

2. Let $(X,\mathcal{A},\mu)$ be a measure space. Let p,q, and r be positive numbers such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $f\in L^p(\mu), g\in L^q(\mu)$, and let $h=fg$. Prove that $h\in L^r(\mu)$ and

$$\|h\|_r \le \|f\|_p\cdot\|g\|_q.$$

3. Suppose $f:[0,1]\to\mathbb{R}$ is increasing. Suppose

$$\int_{[0,1]} f'\,dm = f(1)-f(0).$$

Prove that $f\in AC([0,1])$; that is, f is absolutely continuous on $[0,1]$.

4. For $(x,y)\in[0,1]\times[0,1]$, let

$$f(x,y)=\sum_{n=1}^{\infty}\sqrt{n}[(\sin(xy)]^n.$$

(A) Show that the series defining $f(x,y)$ converges for every $(x,y)\in[0,1]\times[0,1]$. You may assume basic calculus facts about series.

(B) Determine whether f is Lebesgue integrable on $[0,1]\times[0,1]$.

5. Evaluate, with proof,

$$\int_0^\infty \frac{\arctan(\pi x)-\arctan x}{x}\,dm(x).$$

Hint: Recall that $\frac{d}{dt}\arctan t=\frac{1}{1+t^2}$.

6. Let $\Omega = \{z : |z| < 1 \text{ and } \mathrm{Re}(z) > 1/2\}$. Explicitly give a conformal mapping ϕ of Ω onto the unit disc $\mathbb{D}$. (A sequence of explicit analytic functions whose composition gives ϕ will suffice.)

7. Let f and g be analytic in $\mathbb{C}$ except for isolated singularities with $|f(z)| \leq |g(z)|$ wherever both are defined. Show that $f = cg$ where c is a constant.

8. Let G be a region in $\mathbb{C}$, and suppose that for each $n \in \mathbb{N}$, f_n is analytic on G and never vanishes. Prove that if f_n converges to f uniformly on compact sets, then either $f \equiv 0$ or f never vanishes.

9. Let f be analytic in an open set which contains the closed unit disc $\overline{\mathbb{D}}$, and assume $M := \max\{\mathrm{Re}\, f(z) : |z| = 1\} \geq 0$. Prove that for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1+|z|}{1-|z|}\left[M + |f(0)|\right].$$

Analysis Prelim, January 2014

1. Use the residue theorem to calculate $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1} dx$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E_n \in \mathcal{M}$ $(n \geq 1)$ be measurable sets with $\mu(\cup_{n\geq 1} E_n) < \infty$ and $\lim_{n\to\infty} \mu(E_n) = c \in [0, \infty)$. Show that the set A of points that belong to infinitely many E_n is measurable and $\mu(A) \geq c$.

3. Let $\Omega = \{z \in \mathbb{C} : \text{Re}z > 0 \text{ and } \text{Im}z > 0\}$ and let

$$\mathcal{F} = \{f : \mathbb{D} \to \Omega : f \text{ analytic}, f(0) = e^{i\frac{\pi}{4}}\}.$$

Determine

$$\sup\{|f'(0)| : f \in \mathcal{F}\}$$

and then determine all functions in $\mathcal{F}$ for which the sup is attained.

4. Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous functions with $g \in L^1([0, \infty))$ and such that there exists $c > 0$ with $|\int_0^x f(t)dt| \leq c$ for all $x \geq 0$. Show that the following limit exists:

$$\lim_{M\to\infty} \int_0^M f(t) \left(\int_t^\infty g(x)dx \right) dt$$

5. (a). Determine the Laurent series of the function

$$f(z) = \frac{1}{(z-1)^2(z+1)^2}$$

which is valid in the annulus $1 < |z| < 2$.

(b). Compute $\int_\gamma f(z)dz$, where γ is the circle of center 1 and radius 1, with counterclockwise orientation.

6. Let (X, μ) be a complete measure space and let $f \in L^2(X, \mu)$. Show that the set

$$\{p \in [1, \infty) : f \in L^p(X, \mu)\}$$

is an interval (possibly degenerate).

7. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that the function

$$g(x, y) = |f(x + iy)|$$

is integrable on $\mathbb{R}^2$. Show that f is identically 0.

8. Let h be a bounded Lebesgue measurable function on $\mathbb{R}$ such that $\lim_{n\to\infty} \int_E h(nx)dx = 0$, for every Lebesgue measurable set E of finite

measure. Show that for every function f Lebesgue integrable on $\mathbb{R}$ we have:

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)h(nx)dx = 0.$$

9. Determine the number of zeroes of the polynomial $2z^5-6z^2+z+1$ in the annulus $1 < |z| < 2$.

University of Tennessee

Analysis Preliminary Examination
August 2013

1. Find all constants $K > 0$ for which the following holds:
If $(X, \mathcal{M}, \mu)$ is any positive measure space and if $f : X \to \mathbb{R}$ is any integrable function satisfying $|\int_E f d\mu| < K$ for all $E \in \mathcal{M}$, then $||f||_1 < 1$.

2. Determine if there exists a closed curve γ in $\mathbb{C} \setminus \{0, 1\}$ such that

$$\int_\gamma \frac{5z-3}{z^2-z} dz = 2\pi i$$

Prove your answer.

3. Find, with proof, the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x^n)}{x^n} dx$$

4. Let $\mathbb{D}$ denote the open unit disk and let $\Omega = \{x + iy : x, y > 0\} \cap \mathbb{D}$. Determine a conformal map of Ω onto $\mathbb{D}$. You may give your answer as the composition of some functions.

5. Let m denote the Lebesgue measure and let $f_n \in L^2([0,1], m)$ be a sequence of functions, such that f_n converges pointwise to $f \in L^2([0,1], m)$. Assume there is $M > 0$ such that $||f_n||_2 \leq M$ for all n. Prove that for all $g \in L^2([0,1])$ we have

$$\lim_{n\to\infty} \int_{[0,1]} f_n g dm = \int_{[0,1]} f g dm$$

6. Let $\mathbb{D}$ denote the open unit disk, and let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. If $f(\frac{1}{2}) = 0$, find the maximum possible value of $|f(0)|$, and the functions f that attain this value. Justify your answer.

7. Find all entire functions f such that $|f(z)| \geq |z^2 - z|$ for all $z \in \mathbb{C}$. Justify your answer.

8. Let μ and ν be finite positive measures on the measurable space $(X, \mathcal{M})$. Show that there exist nonnegative, measurable functions $f, g : X \to \mathbb{R}$ with $f + g = 1$, such that for all E in $\mathcal{M}$

$$\int_E f d\mu = \int_E g d\nu$$

9. Let $(f_n)_{n\geq 1}$ be a sequence of functions which are analytic on the domain $G \subset \mathbb{C}$. Assume that (f_n) converges uniformly on G to a function f which is not identically zero on G. Show that $a \in G$ is a zero of f if and only if it is a limit point of zeroes of the f_n, $n = 1, 2, 3, \ldots$

Analysis Prelim, January 2013

1. Let $(X, \mathcal{M}, \mu)$ be a measure space, with $\mu(X) = \infty$. Show that $(X, \mathcal{M}, \mu)$ is $\sigma-$finite if and only if there exists a measurable function $f : X \to (0, \infty)$ such that $\int_X f d\mu = 1$.

2. Determine the radius of convergence of the power series of $\frac{2}{e^z + e^{-z}}$. Justify your answer.

3. With proof determine

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n dx.$$

4. Explicitly determine a 1-1 analytic function from

$$G = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - 1/2| > 1/2\}$$

onto the open unit disc. It will be sufficient to explicitly give a sequence of maps whose composition gives the desired map.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(\mu)$. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $E \in \mathcal{M}$ with $\mu(E) < \delta$ then $\int_E |f| d\mu < \varepsilon$.

6. Use the Residue Theorem to calculate

$$\int_0^{2\pi} \cos^{2n}(\theta) \frac{d\theta}{2\pi}.$$

Justify your work.

7. Let f_n, f be positive, integrable functions on the measure space $(X, \mathcal{M}, \mu)$. Assume that $f_n \to f$ pointwise and $\int_X f_n d\mu \to \int_X f d\mu < \infty$. Show that $\int_E f_n d\mu \to \int_E f d\mu$ for all $E \in \mathcal{M}$.

8. Let $\mathbb{D}$ denote the open unit disc, and let f be an analytic function that takes $\mathbb{D}$ into $\mathbb{D}$. Show that for all $z, w \in \mathbb{D}$ $z \neq w$ we have

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \le \left| \frac{z - w}{1 - \overline{w} z} \right|.$$

9. Let μ, ν be finite measures on the measurable space $(X, \mathcal{M})$, with $\nu << \mu$. Let $\lambda = \mu + \nu$ and let $f = \frac{d\nu}{d\lambda}$. Show that $0 \le f < 1$ (μ a.e.) and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

Analysis Prelim, August 2012

1. Let $f(z) = \frac{\cos 2z}{\log(1+3z)}$.

(a) Determine the nature of the singularity that f has at 0 and find its singular part at 0.

(b) Consider the Laurent series of f at 0. Determine the largest set where this series converges.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $A_n \in \mathcal{M}$ $(n \geq 1)$ be a sequence of measurable sets. Let $A = \{x \in X \text{ such that } x \text{ belongs to infinitely many } A_n\}$.

(a) Write a formula for A in terms of the sets A_n.

(b) Show that if $\sum_{n \geq 1} \mu(A_n) < \infty$, then $\mu(A) = 0$.

3. Calculate $\int_0^{2\pi} \frac{d\theta}{k+\cos(\theta)}$ for all reals $k > 1$.

4. Let (X, Σ, μ) be a finite measure space, and let $f_n, f \in L^1(\mu)$ such that $f_n(x) \to f(x)$ a.e. $[\mu]$. Suppose that the sequence $\{f_n\}$ also satisfies the following condition:

For every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $E \in \Sigma$ with $\mu(E) < \delta$, then $\int_E |f_n| d\mu < \varepsilon$ for all n.

Show that $\int_X |f_n - f| d\mu \to 0$ as $n \to \infty$.

5. Let $\Omega = \{z \in \mathbb{C} : 0 < \text{Re } z < 1\}$. Sketch Ω and explicitly give a conformal mapping f of Ω onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. (A sequence of explicit analytic functions whose composition gives f will suffice.)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable such that $f, f' \in L^1(\mathbb{R})$ (Lebesgue measure). Show that $f(x) \to 0$ as $x \to \infty$.

7. Let $\mathbb{D}$ denote the open unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let

$$\mathcal{F} = \{f : \mathbb{D} \to \mathbb{D} : f \text{ is analytic and } f(0) = 0\}.$$

Show that there is an $f_0 \in \mathcal{F}$ such that

$$\int_{-1/2}^{1/2} f_0(x)dx = \sup\{\left|\int_{-1/2}^{1/2} f(x)dx\right| : f \in \mathcal{F}\}.$$

8. Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces, let μ be a σ-finite measure on $(X, \mathcal{S})$, and let λ, ν be σ-finite measures on $(Y, \mathcal{T})$.

Show:

(a) If $\lambda << \nu$, then $\lambda \times \mu << \nu \times \mu$.

(b) If $\lambda \perp \nu$, then $\lambda \times \mu \perp \nu \times \mu$.

9. Let f be an entire function such that $\overline{B(0,1)}$ is contained in $f(\mathbb{C})$. Let V be a component of $f^{-1}(B(0,1))$. Show that V is simply connected.

Analysis Preliminary Examination, January 2012

1. Explicitly determine a 1-1 analytic function from $U = \mathbb{D} \setminus [0, 1)$ onto the unit disc $\mathbb{D}$.

2. With proof determine

$$\lim_{n\to\infty} \int_0^\infty \int_0^1 \frac{y}{\sqrt{x} + ny^3} dxdy.$$

3. Let $\Omega \subseteq \mathbb{C}$ be a region and $f_n : \Omega \to \mathbb{C}$ be analytic functions that are 1-1 and that converge locally uniformly to a non-constant function f on Ω. Show that f is 1-1.

4. Let μ be a positive finite regular measure on the Borel sets of $\mathbb{R}^k$. Show that for any $g \in L^\infty(\mu)$ there is a sequence of compactly supported continuous functions g_n on $\mathbb{R}^k$ such that for all $f \in L^1(\mu)$ we have

$$\int_{\mathbb{R}^k} g_n f d\mu \to \int_{\mathbb{R}^k} g f d\mu \text{ as } n \to \infty.$$

5. Let $f \in L^1([0, \infty))$ (Lebesgue measure). Show that

$$\int_0^\infty e^{-zt} f(t) dt$$

defines an analytic function in $\{z \in \mathbb{C} : \text{ Re } z > 0\}$.

6. Let $M > 0$ and let $g : [0, 1] \to [0, \infty]$ be Borel measurable such that

$$\int_x^y g(t)dt \leq M(y - x) \text{ for all } x, y \in [0, 1], x < y.$$

Show that $g(x) \leq M$ a.e. $[m]$.

7. Show that for every $\varepsilon > 0$, there is a $\delta > 0$ with the following property: whenever $f : \mathbb{D} \to \mathbb{C}$ is analytic such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and $|f(x)| \leq \delta$ for all $-1/2 \leq x \leq 1/2$, then $|f(1/2i)| \leq \varepsilon$.

8. Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Recall that a sequence of measurable functions $f_n : X \to \mathbb{C}$ is said to converge to 0 in measure, if for every $\varepsilon > 0$ there is an integer N such that

$$\mu(\{x \in X : |f_n(x)| > \varepsilon\}) < \varepsilon \quad \text{for all } n \geq N.$$

Let $f_n, g \in L^1(\mu)$ such that

(1) $f_n \to 0$ in measure and

(2) $|f_n(x)| \leq |g(x)|$ for all $n \in \mathbb{N}$ and $x \in X$.

Show that $\int_X |f_n| d\mu \to 0$.

Analysis Preliminary Exam

August 2011

In the following, $\mathbb{C}$ is the complex plane, $\mathbb{R}$ the real line, and $\mathbb{N}$ the natural numbers.

(1) Show that the containments $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$ both fail but that $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ is true. (The measure involved here is Lebesgue measure on $\mathbb{R}$).

(2) Sketch the region $\Omega = \{z : |z| > 1 \text{ and } 0 < \operatorname{Arg}(z) < \frac{\pi}{3}\}$ and give a conformal map f carrying Ω one-to-one onto the unit disc $\mathbb{D} = \{z : |z| < 1\}$. (Note: a sequence of explicit analytic functions whose composition gives f will suffice.)

(3) Let $(X, \mathcal{M}, \mu)$ be a positive measure space. Prove that the following statements are equivalent:

(A) There exists a μ-integrable real function f on X such that f is strictly positive at each point (that is, such that $f(x) > 0$ for each $x \in X$).

(B) μ is σ-finite (that is, there exists μ-measurable sets $\{X_n\}_{n=1}^\infty$ such that $X = \cup_{n=1}^\infty X_n$ and $\mu(X_n) < \infty$ for each n).

(4) Prove that $\sum_{n=1}^\infty \frac{1}{(z-n)^2}$ defines an analytic function in $\mathbb{C} \setminus \mathbb{N}$.

(5) Let (X, μ) be a finite positive measure space and $f \in L^\infty(X, \mu)$, $||f||_\infty > 0$. Let $c_n = \int_X |f|^n d\mu$. Show that $\frac{c_{n+1}}{c_n}$ converges to $||f||_\infty$, as $n \to \infty$.

(6) Let $\Omega = \mathbb{C} \setminus [-1, 1]$ (that is, $\mathbb{C}$ with the real interval $[-1, 1]$ removed) and for $z \in \Omega$ define $f(z) = \frac{1}{z-1} - \frac{1}{z+1} = \frac{2}{z^2-1}$.

(a) Show that for every closed curve γ in Ω we have $\int_\gamma f(z)dz = 0$

(b) Prove that there is an $F \in H(\Omega)$ such that $F' = f$.
Hint: Think of the proof of one of the equivalences in the Riemann mapping theorem.

(7) Let f be analytic and bounded in the right half plane $\{z : \Re z > 0\}$ and suppose $f(t) \longrightarrow 0$ for $t > 0$ and $t \downarrow 0$. Prove that for any z_0 with $\Re z_0 > 0$, we likewise have $f(tz_0) \longrightarrow 0$ as $t \downarrow 0$.

(8) Let $\mathcal{M}$ denote the Lebesgue measurable subsets of $[0, 1]$, and let m denote Lebesgue measure on $[0, 1]$. Suppose μ is a (positive) measure on $([0, 1], \mathcal{M})$ and $\mu(\{0\}) = 0$. Define

$$f(x) = \mu([0, x]),$$

for $0 \leq x \leq 1$. Suppose that f is absolutely continuous on $[0, 1]$.

(a) Prove that $\mu << m$ (that is, μ is absolutely continuous with respect to m).

(b) Prove that $\mu(E) = \int_E f' \, dm$, for all $E \in \mathcal{M}$.

(9) Prove that there exists no one-to-one analytic function mapping the annulus $\{0 < |z| < 1\}$ onto the annulus $\{1 < |z| < 2\}$.

Analysis Prelim, January 2011

$\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disc in the complex plane $\mathbb{C}$.

1. Let $a \in \mathbb{C}$ and let f, g be analytic in a neighborhood of a. Suppose that g has a simple zero at a and that $f(a) \neq 0$. Show that $\mathrm{Res}_a(\frac{f}{g}) = \frac{f(a)}{g'(a)}$.

2. If $f \in L^1(0,\infty)$ be real-valued. Show that

$$\lim_{t\to\infty} t \int_0^\infty \sin\left(\frac{f(tx)}{t}\right) f(tx)dx$$

exists and find its value.

3. Let $U = \{z \in \mathbb{D} : \mathrm{Im} z > 0\}$. Explicitly construct a 1-1 analytic map from U onto $\mathbb{D}$.

4. Let $f_n \in L^2[0,1]$ with $\|f_n\|_{L^2} \leq 1$ for each $n \in \mathbb{N}$ and $f_n(x) \to 0$ a.e. as $n \to \infty$.

Show that $\|f_n\|_{L^1} \to 0$ as $n \to \infty$.

Suggestion: First show that for all $f \in L^2$

$$\int_{|f|\geq N} |f|dm \leq \frac{1}{N}\|f\|_{L^2}^2.$$

5. Show that there is no holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that $|f(z)| \to \infty$ as $|z| \to 1$.

6. Suppose $f \in L^\infty([0,1])$. Show that $\|f\|_{L^p} \to \|f\|_\infty$ as $p \to \infty$.

7. Let $\mathcal{F}$ be a normal family of entire functions and define for $n = 0, 1, 2, \ldots$

$$A_n = \sup\{|a_n| : f(z) = \sum_{j=0}^\infty a_j z^j, f \in \mathcal{F}\}.$$

Show that $\sum_{n=0}^\infty A_n r^n < \infty$ for all $r > 0$.

8. Let $f \in L^1(0,\infty)$. For $x \in [0,\infty)$ set $g(x) = \int_0^x f(t)dt$. Assume that $g \in L^1(0,\infty)$.

(a) Show that g is continuous on $[0,\infty)$.

(b) Show that $g(x) \to 0$ as $x \to \infty$.

9. Let D be an open domain in $\mathbb{C}$, containing the unit disc. Let $f : D \to \mathbb{C}$ be analytic. If $|f(0)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$, show that f is constant.

Analysis Prelim, August 2010

$\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disc in the complex plane $\mathbb{C}$.

1. Let $U = \mathbb{C} \setminus [-1, 1]$. Explicitly construct a bounded, non-constant, analytic function on U.

2. Find (with proof): $\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n dx$.

3. Use the Residue Theorem to calculate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx.$$

Explain all the necessary estimates.

4. Let f be an L^1 function on $\mathbb{R}$ (with respect to Lebesgue measure). Show that $g(x) = \sum_{n=1}^{\infty} f(n^2 x)$ converges a.e. and also show that $g \in L^1(\mathbb{R})$.

5. Suppose that f is holomorphic in an open set containing the closure of $\mathbb{D}$. Show $\partial f(\mathbb{D}) \subseteq f(\partial \mathbb{D})$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_1, f_2, f_3, \ldots$ be a sequence of functions in $L^1(X, \mathcal{M}, \mu)$ converging pointwise to $f \in L^1(X, \mathcal{M}, \mu)$.

Show that f_n converges to f in $\|\cdot\|_1$ (as $n \to \infty$) if and only if $\|f_n\|_1$ converges to $\|f\|_1$ (as $n \to \infty$).

7. Let $K = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \cdots\}$ and let $f \in \mathrm{Hol}(\mathbb{D} \setminus K)$.

Show that if f has a pole at each point $\frac{1}{n}$, $n = 1, 2, \cdots$, then the range of f is dense in $\mathbb{C}$.

8. Let μ, σ be regular finite positive Borel measures on $[0, 1]$. For $g \in C[0, 1]$ define $L(g) = \int_0^1 g\,d\sigma$ and assume that

$$|L(g)|^2 \le \int_{[0,1]} |g|^2 d\mu \quad \text{for every } g \in C[0,1]$$

Show that there is $h \in L^2(\mu)$ such that $\sigma(A) = \int_A h\,d\mu$ for every Borel set A.

9. Let $f, g \in \mathrm{Hol}(\mathbb{D})$ be $1-1$ with $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Show that $|f'(0)| \le |g'(0)|$.