Analysis Diagnostic Exam August 16, 2021

NAME: ______________________________

#1.) ____/10 #2.) ____/10 #3.) ____/10 #4.) ____/10 #5.) ____/10 #6.) ____/10

#7.) ____/10 #8.) ____/10 Total: ______/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) Suppose X and Y are sets and $f : X \to Y$ is a function. Suppose A and B are subsets of X.

(a) (5 points) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) (5 points) If f is $1-1$, prove that $f(A) \cap f(B) \subseteq f(A \cap B)$.

2.) (10 points) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, with $s = \sup A$. Let

$$B = \{a^3 : a \in A\}.$$

Prove that B is bounded above and $\sup B = s^3$. You can assume the elementary properties that every real number has a unique cube root, and, if $x, y \in \mathbb{R}$, then $x < y$ if and only if $x^3 < y^3$.

3.) (10 points) Suppose E is dense in $\mathbb{R}$ (that is, $\overline{E} = \mathbb{R}$) and O is an open subset of $\mathbb{R}$. Prove that

$$O \subseteq \overline{E \cap O}.$$

Here, for any set A, $\overline{A}$ denotes the closure of A.

4.) (10 points) Suppose that (s_n) and (t_n) are convergent sequences of real numbers, with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, for some $s, t \in \mathbb{R}$. Define a sequence (a_n) by

$$(a_1, a_2, a_3, a_4, a_5, a_6, \dots) = (s_1, t_2, s_3, t_4, s_5, t_6, \dots),$$

i.e., $a_n = s_n$ if n is odd, and $a_n = t_n$ if n is even. Prove that (a_n) converges if and only if $s = t$.

5.) (10 points) Suppose $A \subseteq \mathbb{R}$ is non-empty and compact. Prove that

$$-A = \{-a : a \in A\}$$

is compact.

6.) (10 points) Using the $\epsilon - \delta$ definition of limits (and not using any theorems about limits that follow), prove that

$$\lim_{x \to 1} \frac{3x^2 - 1}{x + 1} = 1$$

7.) (10 points) Suppose $a, b, c \in \mathbb{R}$ with $a < b < c$. Suppose f is continuous on the interval (a, c) and differentiable on $(a, b) \cup (b, c)$. Suppose $\lim_{x \to b} f'(x)$ exists. Prove that f is differentiable at b and $f'(b) = \lim_{x \to b} f'(x)$.

8.) (10 points) Suppose $a, b \in \mathbb{R}$ and $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Prove that f^2 is Riemann integrable on $[a, b]$.