Analysis Diagnostic Exam Sample Exercises

A) logic, proofs, induction

A1. Let p and q be statements (i.e., either true or false). Prove that $p \implies q$ is logically equivalent to $\sim (p \wedge \sim q)$. Here $\sim$ denotes the negation of a statement, $p \wedge q$ is the statement that p and q are true, and "logically equivalent" means that one is true if and only if the other is true.

A2. Let p and q be statements. Prove that $p \implies q$ is logically equivalent (defined in A1) to $\sim q \implies \sim p$, where $\sim q$ is the negation of q and $\sim p$ is the negation of p.

A3. Let p, q, and r be statements. Prove that $p \implies (q \vee r)$ holds if and only if $(p \wedge \sim q) \implies r$. Here $\sim$ and $\wedge$ are as in A1, and $q \vee r$ means that q is true or r is true (or both).

A4. Let p, q, and r be statements. Prove that $p \vee q \implies r$ holds if and only if $(p \implies r) \wedge (q \implies r)$ holds. Here $\vee$ and $\wedge$ are as in A1 and A3.

A5. Prove that there is no rational number r such that $r^2 = 2$.

A6. Prove that there is no rational number r such that $r^2 = 3$.

A7. Prove that there is no rational number r such that $r^3 = 2$.

A8. Prove that $\sum_{k=1}^{n}(2k-1) = n^2$, for each $n \in \mathbb{N}$.

A9. Prove that for any $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

A10. Suppose $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n$ are real numbers. Prove that

$$|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|.$$

B) sets, functions

B1. Let X be a set and let A and B be subsets of X. Define $A^c = X \backslash A = \{x \in X : x \notin A\}$ and similarly for B^c. Prove that $A \subseteq B$ if and only if $B^c \subseteq A^c$.

B2. Suppose X is a set, A is a subset of X, and B_λ is a subset of X, for each λ belonging to some index set Λ. Prove that $A \cap (\cup_{\lambda \in \Lambda} B_\lambda) = \cup_{\lambda \in \Lambda}(A \cap B_\lambda)$ and $A \cup (\cap_{\lambda \in \Lambda} B_\lambda) = \cap_{\lambda \in \Lambda}(A \cup B_\lambda)$.

B3. Let A and B be sets. Prove that $A \cap B = A \setminus (A \setminus B)$, where in general $C \setminus D = \{c \in C : c \notin D\}$.

B4. Let A and B be sets. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

B5. Let $f : X \to Y$ be a function. Let Λ be a set, and for each $\lambda \in \Lambda$, assume A_λ is a subset of X. Prove that $f(\cup_{\lambda \in \Lambda} A_\lambda) = \cup_{\lambda \in \Lambda} f(A_\lambda)$, and $f(\cap_{\lambda \in \Lambda} A_\lambda) \subseteq \cap_{\lambda \in \Lambda} f(A_\lambda)$. Give an example (you can choose X, Y, f, Λ and the sets A_λ) such that $f(\cap_{\lambda \in \Lambda} A_\lambda) \neq \cap_{\lambda \in \Lambda} f(A_\lambda)$.

B6. Let $f : X \to Y$ be a function. Let Λ be a set, and for each $\lambda \in \Lambda$, assume B_λ is a subset of Y. Prove that $f^{-1}(\cup_{\lambda\in\Lambda} B_\lambda) = \cup_{\lambda\in\Lambda} f^{-1}(B_\lambda)$, and $f^{-1}(\cap_{\lambda\in\Lambda} B_\lambda) = \cap_{\lambda\in\Lambda} f^{-1}(B_\lambda)$. Here $f^{-1}(B) = \{x \in X : f(x) \in B\}$ is the inverse image of B; we do not assume that f is $1-1$ or onto.

B7. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$\begin{cases} f(n) = \frac{n}{2} & \text{if } n \text{ is even} \\ f(n) = \frac{-n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Prove that f is a bijection.

B8. For $x \in \mathbb{R}$, prove that $\dfrac{x}{1+|x|} \in (-1, 1)$. Define $f : \mathbb{R} \to (-1, 1)$ by $f(x) = \dfrac{x}{1+|x|}$. Prove that f is a bijection.

B9. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Let $g \circ f : X \to Z$ be the composition of f and g, defined by $g \circ f(x) = g(f(x))$.

(i) Assume f and g are $1-1$ (injective). Prove that $g \circ f$ is $1-1$.

(ii) Assume f and g are onto (surjective). Prove that $g \circ f$ is onto.

B10. Suppose X and Y are sets, $f : X \to Y$ is a function, and $A \subseteq X$.

(i) Prove that $A \subseteq f^{-1}(f(A))$.

(ii) Prove that if f is $1-1$, then $f^{-1}(f(A)) = A$.

(iii) Give an example of X, Y, f, and A such that $f^{-1}(f(A)) \neq A$.

C) properties of $\mathbb{R}$, completeness

C1. Let A and B be subsets of $\mathbb{R}$ which are bounded above. Prove that $A \cup B$ is bounded above and $\sup(A \cup B) = \max(\sup A, \sup B)$.

C2. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above. Let $s \in \mathbb{R}$. Prove that $s = \sup A$ if and only s satisfies both (i) s is an upper bound for A and (ii) given any $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

C3. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above, and let $t \in \mathbb{R}$. Define the translate $A + t$ of A by

$$A + t = \{a + t : a \in A\}.$$

Prove that $A + t$ is bounded above, and $\sup(A + t) = t + \sup A$.

C4. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above, and let $-A = \{-x : x \in A\}$. Prove that $-A$ is bounded below and that $\inf(-A) = -\sup A$.

C5. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $r \in \mathbb{R}$ with $r > 0$. Define

$$rA = \{rx : x \in A\}.$$

Prove that rA is bounded above and $\sup(rA) = r \sup A$.

C6. Let A and B be nonempty, bounded above subsets of $\mathbb{R}$. Define

$$A + B = \{a + b : a \in A, b \in B\}.$$

Prove that $A + B$ is bounded above and $\sup(A + B) = \sup A + \sup B$.

C7. Let $A \subseteq (0, \infty)$ be a nonempty bounded set, and let $B = \left\{\frac{1}{x} : x \in A\right\}$. Prove that $\inf B = \frac{1}{\sup A}$.

C8. Suppose that $a_n, b_n \in \mathbb{R}$ with $a_n \leq b_n$, for each $n \in \mathbb{N}$. Let $I_n = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Suppose $I_{n+1} \subseteq I_n$, for each n. Prove that $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

C9. Use the completeness property of $\mathbb{R}$ (the existence of the supremum of any non-empty set which is bounded above) to prove that there exists $x \in \mathbb{R}$ satsifying $x^2 = 2$.

C10. Use the completeness property of $\mathbb{R}$ to prove that there exists $x \in \mathbb{R}$ satsifying $x^3 = 2$.

D) sequences, limits of sequences

D1. Prove the following directly from the ϵ, N definition of the limit of a sequence:

$$\lim_{n\to\infty} \frac{3n + 2}{5n - 12} = \frac{3}{5}.$$

D2. Prove the following directly from the ϵ, N definition of the limit of a sequence:

$$\lim_{n\to\infty} \frac{n^2}{2n^2 - 25} = \frac{1}{2}.$$

D3. Suppose (x_n) is a sequence of real numbers, $\ell \in \mathbb{R}, \lim_{n\to\infty} x_n = \ell$, and $c \in \mathbb{R}$. Prove that $\lim_{n\to\infty}(cx_n) = c\ell$.

D4. Suppose (x_n) is a convergent sequence of real numbers. Prove that (x_n) is bounded.

D5. (Uniqueness of limits) Suppose (x_n) is a sequence of real numbers, $\ell_1, \ell_2 \in \mathbb{R}$, $\lim_{n\to\infty} x_n = \ell_1$, and $\lim_{n\to\infty} x_n = \ell_2$. Prove that $\ell_1 = \ell_2$.

D6. Suppose (x_n) and (y_n) are sequences of real numbers, $\ell_1, \ell_2 \in \mathbb{R}, \lim_{n\to\infty} x_n = \ell_1$, and $\lim_{n\to\infty} y_n = \ell_2$. Prove that

(i) $\lim_{n\to\infty}(x_n + y_n) = \ell_1 + \ell_2$,

and

(ii) $\lim_{n\to\infty}(x_n y_n) = \ell_1 \ell_2$.

D7. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers which is increasing (i.e., $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$) and bounded above. Prove that (x_n) converges to $\sup\{x_n\}_{n=1}^{\infty}$.

D8. Suppose (x_n) is a sequence of real numbers, $\ell \in \mathbb{R}$, and $\lim_{n\to\infty} x_n = \ell$. Prove that $\lim_{n\to\infty} |x_n| = |\ell|$.

D9. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all $x, y \in \mathbb{R}$, and $f(0) = 0$. Let $x_0 \in \mathbb{R}$ be arbitrary. Define $x_1 = f(x_0), x_2 = f(x_1)$, etc., so that $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n = 0$.

D10. Let $x_1 = 2$. For $n \ge 2$, define x_n recursively by $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$.

(i) Prove that $x_n \ge \sqrt{2}$ for all $n \in \mathbb{N}$.

(ii) Prove that (x_n) converges and find $\lim_{n\to\infty} x_n$.

E) subsequences, Cauchy sequences

E1. Let (x_n) and (y_n) be bounded sequences of real numbers. Prove that $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$. Give an example of bounded sequences (x_n) and (y_n) of real numbers with $\limsup(x_n + y_n) \ne \limsup x_n + \limsup y_n$. Here the lim sup of any bounded sequence (a_n) is defined by $\limsup a_n = \inf_{k\ge1} \sup_{m\ge k} a_m$.

E2. Let (x_n) and (y_n) be sequences of real numbers such that (y_n) is bounded and (x_n) converges to some $x \in \mathbb{R}$. Prove that $\limsup(x_n + y_n) = x + \limsup y_n$.

E3. Let (x_n) be a bounded sequence of real numbers. Suppose (x_n) has a subsequence $(x_{n_k})_{k=1}^{\infty}$ which is convergent. Prove that

$$\liminf x_n \le \lim_{k\to\infty} x_{n_k} \le \limsup x_n.$$

Here $\limsup x_n$ is defined as in E1, and $\liminf x_n = \sup_{k\ge1} \inf_{m\ge k} x_m$.

E4. Let (x_n) be a bounded sequence of real numbers. Prove that there is a subsequence of x_n which converges to $\limsup x_n$.

E5. Let (x_n) be a bounded sequence of real numbers. Prove that (x_n) converges if and only if $\liminf x_n = \limsup x_n$.

E6. Prove that any convergent sequence of real numbers in Cauchy.

E7. Prove that any Cauchy sequence of real numbers is bounded (without assuming the theorem that a Cauchy sequence of real numbers converges).

E8. Prove that any Cauchy sequence of real numbers is convergent (you can use E4 and E7).

E9. Let (x_n) be a bounded sequence of real numbers. Let

$$S = \left\{ x \in \mathbb{R} : \text{ there exists a subsequence } \{x_{n_k}\}_{k=1}^{\infty} \text{ of } (x_n) \text{ such that } \lim_{k\to\infty} x_{n_k} = x \right\};$$

that is, S is the set of subsequential limit points of (x_n). Prove that S is a closed set; that is, if $y_n \in S$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y$, then $y \in S$.

E10. Let (x_n) be a sequence of real numbers, and let $\ell \in \mathbb{R}$. Prove that $\lim_{n\to\infty} x_n = \ell$ if and only if every subsequence of (x_n) has a subsequence converging to ℓ.

F) open and closed sets

F1. Using only the definition of open sets (i.e., $O \subseteq \mathbb{R}$ is open if, for each $x \in O$, there exists $\epsilon > 0$, where ϵ may depend on x, such that $(x-\epsilon, x+\epsilon) \subseteq O$), prove that

(i) an arbitrary union of open sets is open,
and
(ii) a finite intersection of open sets is open.

F2. Using only the definition of closed sets (i.e., $E \subseteq \mathbb{R}$ is closed if, for each sequence (x_n) with $x_n \in E$ for all $n \in \mathbb{N}$, which converges to some $x \in \mathbb{R}$, we have $x \in E$), prove that

(i) an arbitrary intersection of closed sets is closed,
and
(ii) a finite union of closed sets is closed.

F3. Using only the definition of open sets (see F1), prove that the interval $(-1, 1)$ is open.

F4. Give an example of open sets $\{O_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\cap_{n=1}^{\infty} O_n$ is not open. Prove your answer.

F5. Give an example of closed sets $\{E_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\cup_{n=1}^{\infty} E_n$ is not closed. Prove your answer.

F6. Using only the definition of open and closed sets (see F1 and F2 for the definitions), prove that a subset O of $\mathbb{R}$ is open if and only if $E = \mathbb{R} \setminus O$ is closed.

F7. For $E \subseteq \mathbb{R}$, define $\overline{E}$, the closure of E, by

$$\overline{E} = \{x \in \mathbb{R} : \text{ there exists a sequence } (x_n) \subseteq E \text{ such that } \lim_{n\to\infty} x_n = x\}.$$

(i) Prove that $E \subseteq \overline{E}$ and $\overline{E}$ is closed.

(ii) Suppose $F \subseteq \mathbb{R}$, F is closed, and $E \subseteq F$. Prove that $\overline{E} \subseteq F$. Deduce the characterization

$$\overline{E} = \cap\{F : F \subseteq \mathbb{R}, F \text{ is closed, and } E \subseteq F\}.$$

F8. Suppose $A, B \subseteq \mathbb{R}$. Prove that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

F9. Suppose that E_λ is a subset of $\mathbb{R}$ for every $\lambda \in \Lambda$, where Λ is an arbitrary index set.

(i) Prove that $\cup_{\lambda\in\Lambda} \overline{E_\lambda} \subseteq \overline{\cup_{\lambda\in\Lambda} E_\lambda}$.

(ii) Give an example of sets $E_n \subseteq \mathbb{R}$, for $n \in \mathbb{N}$, such that $\cup_{n=1}^{\infty} \overline{E_n} \neq \overline{\cup_{n=1}^{\infty} E_n}$.

F10. Suppose $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. For $x \in \mathbb{R}$, let $d(x) = \inf\{|x - a| : a \in A\}$ (d is the distance from the point x to the set A). Prove that $x \in \overline{A}$ if and only if $d(x) = 0$, where $\overline{A}$ is the closure of A.

G) compact sets

G1. Give an example of an open cover of $[0, 1)$ which has no finite subcover. Prove your answer.

G2. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Give an example of an open cover of A which has no finite subcover. Prove your answer.

G3. Let $K = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Prove directly, without using the Heine-Borel or Bolzano-Weierstrass theorems, that K is compact.

G4. Using only the open cover definition of compactness (and not the Heine-Borel or Bolzano-Weierstrass theorems), prove that the union of two compact subsets of $\mathbb{R}$ is compact.

G5. Using only the open cover definition of compactness (and not the Heine-Borel or Bolzano-Weierstrass theorems), prove that a compact subset of $\mathbb{R}$ is closed.

G6. Using only the open cover definition of compactness (and not the Heine-Borel or Bolzano-Weierstrass theorems), prove that a compact subset of $\mathbb{R}$ is bounded.

G7. Using only the open cover definition of compactness (and not the Heine-Borel or Bolzano-Weierstrass theorems), prove that a closed subset of a compact set is compact.

G8. Suppose $K \subseteq \mathbb{R}$ is compact and non-empty. Show that $\sup K \in K$ and $\inf K \in K$.

G9. Suppose $K_j \subseteq \mathbb{R}$ is compact for each $j \in \mathbb{N}$ and $\cap_{j=1}^{n} K_j \neq \emptyset$ for each $n \in \mathbb{N}$. Prove that $\cap_{j=1}^{\infty} K_j \neq \emptyset$. Give an example of closed sets $E_j \subseteq \mathbb{R}$ such that $\cap_{j=1}^{n} E_j \neq \emptyset$ and $\cap_{j=1}^{\infty} E_j = \emptyset$.

G10. Suppose $A, B \subseteq \mathbb{R}$ are non-empty, with A compact and B closed. If $A \cap B = \emptyset$, prove that there exists $\epsilon > 0$ such that $|a - b| > \epsilon$ for all $A \in A$ and $b \in B$. (Here ϵ is independent of a, b.) Give an example showing that the conclusion fails if A is only assumed to be closed.

H) limits of functions

H1. Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. Give an example of a sequence of points (x_n) with $x_n > 0$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n \to \infty} \sin\left(\frac{1}{x_n}\right) = 0$. How is the existence of that sequence (x_n) consistent with the non-existence of $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$?

H2. Prove (directly from the $\epsilon - \delta$ definition of limits) that $\lim_{x\to 3} x^2 = 9$.

H3. Prove (directly from the $\epsilon - \delta$ definition of limits) that $\lim_{x \to 0} \frac{1}{x+3} = \frac{1}{3}$.

H4. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist in $\mathbb{R}$, for some $c \in \mathbb{R}$. Prove that $\lim_{x\to c}(f + g)(x)$ exists and $\lim_{x\to c}(f +$

$g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

H5. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist in $\mathbb{R}$, for some $c \in \mathbb{R}$. Prove that $\lim_{x\to c}(fg)(x)$ exists and $\lim_{x\to c}(fg)(x) = (\lim_{x\to c} f(x)) \cdot (\lim_{x\to c} g(x))$.

H6. Suppose $f : (-1,1) \to \mathbb{R}$ and $g : (-1,1) \to \mathbb{R}$ are functions such that $\lim_{x\to 0} f(x) = 0$ and g is bounded. Prove that $\lim_{x\to 0}(fg)(x) = 0$.

H7. Suppose $a, b \in \mathbb{R}$ with $a < b, c \in (a,b)$, and $f : (a,b) \to \mathbb{R}$ is a function. Prove that $\lim_{x\to c} f(x) = L$ if and only if: for all sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in (a,b) \setminus \{c\}$ for all $n \in \mathbb{N}$ and satisfying $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = L$.

H8. Suppose $f, g, h : (-1,1) \to \mathbb{R}$ satisfy $f(x) \le g(x) \le h(x)$ for all $x \in (-1,1)$, $\lim_{x\to 0} f(x) = a$ and $\lim_{x\to 0} h(x) = a$, for some $a \in \mathbb{R}$. Prove that $\lim_{x\to 0} g(x) = a$.

H9. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function, and $f(x) \ge 0$ for all $x \in \mathbb{R}$. Suppose $\lim_{x\to c} f(x) = 0$, for some $c \in \mathbb{R}$. Prove that $\lim_{x\to c} \sqrt{f(x)} = 0$.

H10. Suppose $O \subseteq \mathbb{R}$ is a non-empty open set, and $f : O \to \mathbb{R}$ is a function. Suppose $c \in O$ and $\lim_{x\to c} f(x)$ exists, with $\lim_{x\to c} f(x) > 0$. Suppose also that $f(c) > 0$. Prove that there exist $\ell, r > 0$ such that such that $f(x) > \ell$ for all $x \in (c - r, c + r)$.

I) continuity of functions

I1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0. Prove that there exists $\epsilon > 0$ such that f is bounded on $(-\epsilon, \epsilon)$.

I2. Prove directly from the $(\epsilon - \delta)$ definition of continuity that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

I3. Suppose $f : E \to \mathbb{R}$ is a function, where $E \subseteq \mathbb{R}$. Let $x_0 \in E$. Prove that f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences (x_n) contained in E such that $\lim_{n\to\infty} x_n = x_0$.

I4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Prove that f is continuous on $\mathbb{R}$ if and only if $f^{-1}(O)$ is open for every open set $O \subseteq \mathbb{R}$. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a non-empty open set $O \subseteq \mathbb{R}$ such that $f(O)$ is not open in $\mathbb{R}$.

I5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Prove that f is continuous on $\mathbb{R}$ if and only if $f^{-1}(E)$ is closed for every closed set $E \subseteq \mathbb{R}$. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a non-empty closed set $E \subseteq \mathbb{R}$ such that $f(E)$ is not closed in $\mathbb{R}$.

I6. Suppose $K \subseteq \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous. Prove that f attains a maximum on K; that is, there exists a point $x_0 \in K$ such that $f(x) \le f(x_0)$ for all $x \in K$. You can assume the Bolzano-Weierstass theorem.

I7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact. Prove that $f(K)$ is compact.

I8. Suppose $A \subseteq \mathbb{R}$, with $A \neq \emptyset$.

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $f(\overline{A}) \subseteq \overline{f(A)}$.

(ii) Give an example of a nonempty set A and a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\overline{A}) \not\subseteq \overline{f(A)}$.

(iii) Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ and a nonempty set $A \subseteq \mathbb{R}$ such that $f(\overline{A}) \neq \overline{f(A)}$.

I9. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f^3(x)$ is continuous on $\mathbb{R}$. Prove that f is continuous on $\mathbb{R}$. Give an example of $f : \mathbb{R} \to \mathbb{R}$ such that $f^2(x)$ is continuous on $\mathbb{R}$ but f is not continuous on $\mathbb{R}$.

I10. Give an example of a continuous function on $(0, 1)$ and a Cauchy sequence (x_n) in $(0, 1)$ such that $(f(x_n))$ is not a Cauchy sequence in $\mathbb{R}$.

J) uniform continuity

J1. Is $f(x) = x^2$ uniformly continuous on $(0, 1)$? Prove your answer.

J2. Is $f(x) = x^2$ uniformly continuous on $(0, \infty)$? Prove your answer.

J3. Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Prove that f is bounded.

J4. Is $f(x) = \sin(1/x)$ uniformly continuous on $(0, 1)$? Prove your answer.

J5. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous. Prove that $f + g$ is uniformly continuous on $\mathbb{R}$.

J6. Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ which are uniformly continuous, but fg is not uniformly continuous on $\mathbb{R}$. Prove that your answer has the required properties.

J7. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous. Prove that $g \circ f$ is uniformly continuous on $\mathbb{R}$.

J8. Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Prove that there exists a function $g : [0, 1] \to \mathbb{R}$ such that g is continuous and $g(x) = f(x)$ for all $x \in (0, 1)$ (i.e., g is an extension of f).

J9. Let $A \subseteq \mathbb{R}$ with $A \neq \emptyset$. Define $d : \mathbb{R} \to [0, \infty)$ by $d(x) = \inf\{|x - y| : y \in A\}$ (d is the distance to the set A). Prove that d is uniformly continuous on $\mathbb{R}$.

J10. Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous on $(0, 1)$ and (x_n) is a Cauchy sequence in $(0, 1)$. Prove that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

K) the derivative

K1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\ . \end{cases}$ Determine whether f is differentiable at 0. If f is differentiable at 0, determine whether f' is continuous at 0. Prove your answers.

K2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Determine whether f is differentiable at 0. If f is differentiable at 0, determine whether f' is continuous at 0. Prove your answers.

K3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at some point $c \in \mathbb{R}$. Prove that f is continuous at c.

K4. Give and example of a function $f : \mathbb{R} \to \mathbb{R}$ such that (i) f is differentiable at 0 and (ii) f is not continuous at all $x \neq 0$.

K5. Suppose $a, b \in \mathbb{R}$ with $a < b$, and suppose $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Suppose f has a local minimum at a point $c \in (a, b)$ (that is, for some $\epsilon > 0$, we have $f(x) \geq f(c)$ for all $x \in (c - \epsilon, c + \epsilon)$). Prove that $f'(c) = 0$.

K6. Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable on (a, b). Prove that if $f(a) = f(b) = 0$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

K7. Suppose $a, b \in \mathbb{R}$ with $a < b$, and suppose $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$. Prove that f is strictly increasing on (a, b); that is, for $a < c < d < b$, we have $f(c) < f(d)$.

K8. Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose f, f', and f'' exist on (a, b), Prove that if f has a local maximum at $c \in (a, b)$ (that is, for some $\epsilon > 0$, we have $f(x) \leq f(c)$ for all $x \in (c - \epsilon, c + \epsilon)$), then $f''(c) \leq 0$.

K9. Suppose $a, b, c \in \mathbb{R}$ with $a < b < c$. Suppose f is continuous on the interval (a, c) and differentiable on $(a, b) \cup (b, c)$. Suppose $\lim_{x \to b} f'(x)$ exists. Prove that f is differentiable at b and $f'(b) = \lim_{x \to b} f'(x)$.

K10. Give an example of a function $f : [-1, 1] \to \mathbb{R}$ such that f is continuous on $[-1, 1]$, f is differentiable on $(-1, 1)$, and $f'(0) > 0$, but there is no interval around 0 on which f is nondecreasing.

L) sequences of functions

L1. Define $f_n : (0, 1) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+nx^2}$ for $n \in \mathbb{N}$. Prove that $\lim_{x \to 0^+} \lim_{n \to \infty} f_n(x) = +\infty$ and $\lim_{n \to \infty} \lim_{x \to 0^+} f_n(x) = 0$.

L2. Give an example of a sequence of continuous functions (f_n) defined on $[0, 1]$ which are uniformly bounded (i.e., there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$) such that $\lim_{n \to \infty} f_n(x) = f(x)$ exists for each $x \in [0, 1]$, but f is not continuous on $[0, 1]$.

L3. Give an example of a sequence of continuous functions f_n defined on $[0, 1]$ such that $\lim_{n \to \infty} f_n(x) = f(x)$ exists for each $x \in [0, 1]$, and a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [0, 1]$ for each $n \in \mathbb{N}$ such that $\lim_{n \to \infty} x_n = x$ exists in $\mathbb{R}$ and $\lim_{n \to \infty} f_n(x_n)$ exists, but $\lim_{n \to \infty} f_n(x_n) \neq f(x)$.

L4. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ satisfies $|f_n(x) - f_n(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$, for each $n \in \mathbb{N}$, and suppose $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in \mathbb{R}$. Prove that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

L5. Give an example of a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that each f_n is differentiable, f is differentiable, and $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$, but there exists $x_0 \in \mathbb{R}$ such that $f_n'(x_0)$ does not converge to $f'(x_0)$. $(\frac{1}{n}\sin(ne^x))$

L6. Give an example of a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\lim_{n\to\infty} \lim_{x\to\infty} f_n(x) \neq \lim_{x\to\infty} \lim_{n\to\infty} f_n(x)$, where all of the limits exist in $\mathbb{R}$.

L7. Show that there exists a sequence of functions (f_n) on $\mathbb{R}$ such that each f_n is continuous on $\mathbb{R} \setminus E_n$, where each E_n is a finite set, and $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for every $x \in \mathbb{R}$, but f is discontinuous at every point of $\mathbb{R}$.

L8. Suppose $f_n : (0,1) \to \mathbb{R}$ is increasing (i.e., $f_n(x) \leq f_n(y)$ for all $x < y$ with $x, y \in (0,1)$). Suppose $f(x) = \lim_{n\to\infty} f_n(x) < \infty$ for every $x \in (0,1)$. Prove that $f(x) \leq f(y)$ for all $x < y$ with $x, y \in (0,1)$. Give an example where each f_n is strictly increasing (i.e., $f_n(x) < f_n(y)$ for all $x, y \in (0,1)$ with $x < y$), but f is not strictly increasing.

L9. Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists (in $\mathbb{R}$) for each $x \in [0,1]$, and such that f is unbounded on $[0,1]$.

L10. Give an example of a sequence of functions $f_n : [0,1] \to \mathbb{R}$ such that for each $n \in \mathbb{N}$, f_n is continuous and $\int_0^1 f_n(x)\,dx = 1$, but $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$.

M) uniform convergence

M1. For $n \in \mathbb{N}$, let $f_n(x) = \sin\left(\frac{x}{n}\right)$. Does the sequence (f_n) converge uniformly on $[0,1]$? Does (f_n) converge uniformly on $[0,\infty)$? Prove your answers.

M2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. For $n \in \mathbb{N}$, define $f_n(x) = f\left(x + \frac{1}{\sqrt{n}}\right)$.

(i) Prove that if f is uniformly continuous on $\mathbb{R}$, then f_n converges uniformly to f on $\mathbb{R}$.

(ii) Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ such that f_n does not converge uniformly to f. Prove your conclusion.

M3. Suppose $A, B \subseteq \mathbb{R}$, and (f_n) is a sequence of functions with $f_n : A \cup B \to \mathbb{R}$ such that f_n converges uniformly on A to some function f, and f_n converges uniformly on B to f. Prove that f_n converges uniformly to f on $A \cup B$.

M4. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded functions on $\mathbb{R}$ which converges uniformly to a function f on $\mathbb{R}$. Prove that there exists $M < \infty$ such that $|f(x)| \leq M$ and $|f_n(x)| \leq M$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ (where M is independent of x and n).

M5. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions (with $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$) which converges uniformly to a function f on $\mathbb{R}$. If each f_n is continuous at some point $c \in \mathbb{R}$, prove that f is continuous at c.

M6. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions (with $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$) which converges uniformly to a function f on $\mathbb{R}$. If each f_n is uniformly continuous on $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$.

M7. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions (with $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$) which converges uniformly to a function f on $\mathbb{R}$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers and let $x = \lim_{n\to\infty} x_n \in \mathbb{R}$. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

M8. Suppose (f_n) is sequence of functions, all defined on the same non-empty subset S of $\mathbb{R}$. Prove that (f_n) is uniformly Cauchy on S if and only if (f_n) converges uniformly on S. (The sequence (f_n) is said to be *uniformly Cauchy* on S if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_j(x) - f_k(x)| < \epsilon$ for all $j, k > N$ and all $x \in S$. Note that N is independent of x.)

M9. Suppose that for each $j \in \mathbb{N}$, $f_j : [0, 1] \to \mathbb{R}$ satisfies $|f_j(x) - f_j(y)| \leq M|x - y|$, for all $x, y \in [0, 1]$, with M a constant (independent of x, y, and j). Suppose $\lim_{j\to\infty} f_j(x)$ exists (as a real number) for all $x \in [0, 1]$, and let $f(x) = \lim_{j\to\infty} f_j(x)$. Prove that f_j converges to f uniformly on $[0, 1]$.

M10. Suppose that for each $j \in \mathbb{N}$, $f_j : [0, 1] \to \mathbb{R}$ is continuous, with $f_j(x) \leq f_{j+1}(x)$ for each $x \in [0, 1]$ and $j \in \mathbb{N}$. Suppose $\lim_{j\to\infty} f_j(x)$ exists for all $x \in [0, 1]$, and $f(x) = \lim_{j\to\infty} f_j(x)$ is continuous. Prove that f_j converges to f uniformly on $[0, 1]$.

N) Riemann integration

N1. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$ for $0 \leq x \leq 1/2$ and $f(x) = 1$ for $1/2 < x \leq 1$. Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}. \cdots, 1\}$ be the regular partition of size $1/n$ for $[0, 1]$. Compute $U(f, P_n)$ and $L(f, P_n)$, the upper and lower Riemann sums for f on P_n. Deduce that f is Riemann integrable on $[0, 1]$.

N2. Compute $U(f, P_n)$ for $f(x) = x$, where P_n is the uniform grid in N1. Take the limit to obtain $\int_0^1 x\,dx$. You may use the formula $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$.

N3. Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is increasing: if $x, y \in [a, b]$ and $x < y$ then $f(x) \leq f(y)$. Prove that f is Riemann integrable on $[a, b]$.

N4. For $n \in \mathbb{N}$, let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n + 1)$.

(i) Prove that $s_n \leq 1 - \frac{1}{n+1}$ for all $n \in \mathbb{N}$.

(ii) Prove that $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.

N5. Show that there exists a sequence of Riemann integrable functions (f_n) on $[0, 1]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists in $\mathbb{R}$ for each $x \in [0, 1]$ but f is not Riemann integrable on $[0, 1]$.

N6. Suppose $f_n : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, for each $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$ with $a < b$. Suppose f_n converges uniformly to some function $f : a, b \to \mathbb{R}$ on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

N7. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of Riemann integrable functions on an interval $[a, b]$ (here $a, b \in \mathbb{R}$ with $a < b$) such that f_n converges uniformly to a function f on $[a, b]$. Prove that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. You can assume the result (see N6) that f is Riemann integrable on $[a, b]$.

N8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and satisfies $f'(x) = \cos(1 + x + f(x))$ for all $x \in \mathbb{R}$. Prove that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

N9. Suppose $f : [0, 1] \to \mathbb{R}$ is Riemann integrable (in particular, f is bounded). Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = \int_0^x f(t)\,dt$.

(i) Give an example of an f as stated, such that g is not differentiable at $x = 1/2$.

(ii) Prove that there exists an $M \in [0, \infty)$ such that $|g(b) - g(a)| \leq M|b - a|$, for all $a, b \in [0, 1]$. Here M depends on f but not on a or b.

N10. Suppose (f_n) is a sequence of functions which are differentiable on $(-1, 1)$ such that f_n converges uniformly to a function f on $(-1, 1)$. Suppose also that f_n' converges uniformly on $(-1, 1)$ to a function g. Prove that f is differentiable on $(-1, 1)$ and $f' = g$.