Numerical Analysis Preliminary Examination

August 14, 2023

Instructions

1. There are nine problems. Read each problem carefully.
2. Write your solutions on one side of the provided solution paper, one side only, starting a new problem at the top of a new sheet of paper.
3. At the top of each solution page, write (a) your full name and (b) the problem number followed by the page number, counting from one at the start of each new problem. For example, page three of the fourth problem would be numbered 4-3, and, page two of the seventh problem, 7-2.
4. Hand in only your solution pages, un-stapled and in numerical order. This five-page exam packet is yours to keep.
5. You must show your work to receive credit.
6. If, in the solution of a problem, you need to invoke a known fact — that is, a theorem/lemma/proposition, et cetera — you must clearly state the assumptions and conclusions of the cited fact. In addition, you must explicitly verify that all the assumptions are satisfied.
7. If you believe a problem has a typo, missing conditions, or it can be interpreted in several ways, please clearly indicate so in your work. In this case, fix the problem in a way that it does not become trivial.

1 Numerical Linear Algebra

1. The purpose of this problem is to provide a proof of existence of the SVD. For this reason, you *cannot use* the SVD to solve it.

   Let $\mathsf{A} \in \mathbb{C}^{m\times n}$ with $m \geq n$.

   (a) Show that both $\mathsf{A}\mathsf{A}^H$ and $\mathsf{A}^H\mathsf{A}$ are Hermitian and positive semi-definite. Therefore they admit the decompositions

   $$\mathsf{A}\mathsf{A}^H = \mathsf{U}\Sigma\Sigma^T\mathsf{U}^H, \qquad \mathsf{A}^H\mathsf{A} = \mathsf{V}\Sigma^T\Sigma\mathsf{V}^H,$$

   where $\mathsf{U} \in \mathbb{C}^{m\times m}$ and $\mathsf{V} \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ is diagonal and has nonnegative entries. Notice that this must be the same matrix in both decompositions.

   (b) From the previous two decompositions show that $\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^H$. **Hint:** Show that the columns of U are eigenvectors of $\mathsf{A}\mathsf{A}^H$.

2. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$. Let $\mathsf{A} = [a_{i,j}]$ be strictly diagonally dominant, of dominance $\delta > 0$, that is,

   $$|a_{i,i}| \geq \delta + \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|, \quad i = 1, \dots, n.$$

   Show that A is nonsingular and that

   $$\|\mathsf{A}^{-1}\|_\infty \leq \delta^{-1}.$$

3. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be nonsingular, $\boldsymbol{f} \in \mathbb{C}^n$, and $\boldsymbol{x} \in \mathbb{C}^n$ be such that $\mathsf{A}^H\mathsf{A}\boldsymbol{x} = \mathsf{A}^H\boldsymbol{f}$. Let $\mathsf{P} \in \mathbb{C}^{n\times n}$ be such that $2\|\mathsf{P}\|_2 \leq \sigma_n(\mathsf{A})^2$, where $\sigma_n(\mathsf{A})$ is the smallest singular value of A.

   (a) Show that $\mathsf{A}^H\mathsf{A} + \mathsf{P}$ is nonsingular.

   (b) Let $\boldsymbol{y} \in \mathbb{C}^n$ be such that $(\mathsf{A}^H\mathsf{A} + \mathsf{P})\boldsymbol{y} = \mathsf{A}^H\boldsymbol{f}$. Define, for $\boldsymbol{z} \in \mathbb{C}^n$, $\boldsymbol{r}(\boldsymbol{z}) = \boldsymbol{f} - \mathsf{A}\boldsymbol{z}$. Show that

   $$\boldsymbol{r}(\boldsymbol{y}) - \boldsymbol{r}(\boldsymbol{x}) = \mathsf{A}(\mathsf{A}^H\mathsf{A} + \mathsf{P})^{-1}\mathsf{P}\boldsymbol{x},$$
   $$\|\boldsymbol{r}(\boldsymbol{y}) - \boldsymbol{r}(\boldsymbol{x})\|_2 \leq 2\kappa_2(\mathsf{A})^2 \frac{\|\mathsf{P}\|_2}{\|\mathsf{A}\|_2}\|\boldsymbol{x}\|_2,$$

   where $\kappa_2(\mathsf{A})$ is the spectral condition number of A.

   (c) Let $\boldsymbol{g} \in \mathbb{C}^n$ be such that $\|\boldsymbol{g}\|_2 \leq \|\mathsf{A}^H\|_2\|\boldsymbol{f}\|_2$ and $\boldsymbol{w} \in \mathbb{C}^n$ satisfy $\mathsf{A}^H\mathsf{A}\boldsymbol{w} = \mathsf{A}^H\boldsymbol{f} + \boldsymbol{g}$. Show that

   $$\frac{\|\boldsymbol{x} - \boldsymbol{w}\|_2}{\|\boldsymbol{x}\|_2} \leq \kappa_2(\mathsf{A})^2 \frac{\|\mathsf{A}^H\|_2\|\boldsymbol{f}\|_2}{\|\mathsf{A}^H\boldsymbol{f}\|_2}.$$

2 Numerical Solution of Nonlinear Equations

4. Let $a, b \in \mathbb{R}$ with $a < b$. Assume that $f \in C^3([a, b])$ and that $\xi \in (a, b)$ is a root of f of multiplicity 2, that is,
$$f(\xi) = 0 = f'(\xi),$$
but $f''(\xi) \neq 0$. Consider the following *Modified Newton's Method* for finding the value of the root:
$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}.$$
Prove that the method converges locally and *quadratically* to ξ.

3 Numerical Solution of ODEs

5. Consider the BDF2 scheme for approximating the solution to $u'(t) = f(t, u(t))$,
$$\frac{w^{k+2} - \frac{4}{3}w^{k+1} + \frac{1}{3}w^k}{\frac{2}{3}\tau} = f(t_{k+2}, w^{k+2}).$$
Prove that the scheme is root stable, consistent of order 2, and A-stable.

6. Recall that all r-stage Runge-Kutta (RK) schemes are uniquely described by their Butcher tableau
$$\begin{array}{c|c} \boldsymbol{c} & \mathsf{A} \\ \hline & \boldsymbol{b}^\intercal \end{array}.$$
A *diagonally implicit* RK (DIRK) scheme is one for which $\mathsf{A} = [a_{i,j}]_{i,j=1}^r$ is lower triangular.

   (a) Derive all the second order, two-stage, DIRK schemes, such that

      i. The matrix A has positive diagonal entries, that is, $a_{1,1} >$ and $a_{2,2} > 0$.

      ii. The second stage coincides with the solution at the next step, that is, $\boldsymbol{b}^\intercal = [a_{2,1}, a_{2,2}]$.

   Show that if, in addition, we wish that the diagonal entries coincide, that is, $a_{1,1} = a_{2,2}$, then the only possible scheme is the so-called *Alexander's scheme*:
$$\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array} \qquad \gamma = 1 \pm \frac{1}{\sqrt{2}}. \tag{1}$$

   (b) Investigate the A-stability of Alexander's scheme (1) by analyzing the amplification factor
$$g(z) = 1 + z\boldsymbol{b}^\intercal(\mathsf{I} - z\mathsf{A})^{-1}\mathbf{1}.$$

4 Numerical Solution of PDEs

7. Consider the following two point boundary value problem:

$$-(p(x)u')' + r(x)u = f(x), \quad x \in (0,1), \qquad u(0) = u(1) = 0.$$

Assume that $p \in C^1([0,1])$, with $p(x) \geq p_0 > 0$, for all $x \in [0,1]$, $r \in C([0,1])$, with $r(x) \geq 0$, for all $x \in [0,1]$, and $f \in L^2(0,1)$.

(a) Write a weak formulation of this problem, and show that it has a unique solution $u \in H_0^1(0,1)$.

(b) Let $n \in \mathbb{N}$, $n \geq 2$. Consider a uniform mesh of size $h = 1/n$, and a finite element space of piece-wise linear functions. Let u_h denote the finite element solution to the two point boundary value problem. Show that, provided u is smooth enough,

$$\|u' - u_h'\|_{L^2(0,1)} \leq Ch\|u''\|_{L^2(0,1)},$$

for some constant $C > 0$.

(c) Let $n = 3$, $p(x) = 1$, $r(x) = 0$, and $f(x) = x$. Explicitly write the ensuing stiffness matrix and load vector from the previous item.

8. We say that the eigenvalue problem

$$-u''(x) = \lambda u(x), \quad x \in (0,1), \qquad u(0) = u(1) = 0,$$

has a solution if there are $\lambda \in \mathbb{R}$ and a nonzero function $u \in C^2(0,1) \cap C([0,1])$, for which the equation and boundary conditions hold point-wise.

(a) Show that, for every $m \in \mathbb{N}$, the pair (λ_m, u_m) with

$$\lambda_m = m^2\pi^2, \qquad u_m(x) = \sin\left(\sqrt{\lambda_m}x\right),$$

solves the eigenvalue problem.

(b) Let $N \in \mathbb{N}$, $h = \frac{1}{N} > 0$, and Ω_h be a uniform mesh over $(0,1)$ with mesh size h. $\mathcal{V}_0(\Omega_h)$ denotes the set of all grid functions on Ω_h that vanish at $x = 0$ and $x = 1$. Consider the following finite difference (FD) approximation of the eigenvalue problem: Find $\mu \in \mathbb{R}$ and a nonzero grid function $w \in \mathcal{V}_0(\Omega_h)$ such that

$$-\Delta_h w = \mu w,$$

where

$$\Delta_h w_j = \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2}, \qquad j = 1, 2, \ldots, N-1.$$

For every $m \in \mathbb{N}$, define $w_m \in \mathcal{V}_0(\Omega_h)$ via

$$w_{m,j} = \sin\left(\sqrt{\lambda_m}jh\right).$$

Find μ_m so that the pair (μ_m, w_m) solves the discrete eigenvalue problem.

(c) For $m \in \mathbb{N}$, establish the following error estimate

$$|\lambda_m - \mu_m| \leq m^4 \pi^4 \frac{h^2}{12}.$$

Hint: For every θ, there is ξ with $|\xi| \leq 1$ such that

$$1 - \cos(\theta) = \frac{1}{2}\theta^2 - \frac{1}{24}\xi\theta^4.$$

9. The purpose of this problem is to extend the tools for analyzing schemes to a new spatio-temporal PDE. Specifically, to approximate the following dispersive system

$$\partial_t u + \partial_{xxx} u = 0, \qquad t > 0, \qquad x \in \mathbb{R}$$

let us employ the following scheme

$$\frac{w_j^{k+1} - w_j^k}{\tau} + \frac{w_{j+2}^k - 3w_{j+1}^k + 3w_j^k - w_{j-1}^k}{h^3} = 0.$$

(a) Given $h > 0$, show that, for any $v \in C_b^5(\mathbb{R})$,

$$\frac{v_{j+2} - 3v_{j+1} + 3v_j - v_{j-1}}{h^3} = v'''(x_j) + \frac{h}{2}v^{(4)}(x_j) + \mathcal{O}(h^2),$$

where $x_j = j \cdot h$, for all $j \in \mathbb{Z}$, and $v_j := v(x_j)$.

(b) Assuming that the PDE solution $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is sufficiently smooth, with sufficiently bounded derivatives, prove that the consistency error, $\mathcal{E}[u]$, satisfies

$$\left|\mathcal{E}[u](x_j, t^k)\right| \leq C\left(\tau + h\right),$$

for some $C > 0$ that is independent of τ, h, j, and k.

(c) Let $\nu = \frac{\tau}{h^3}$. Assume that, as τ and h change, ν remains constant. Find the symbol (amplification factor) of the scheme.

January 2023 Numerical Analysis Prelim

1 Numerical Linear Algebra

1. Let $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{n\times n}$ be strictly (row-wise) diagonally dominant, that is,

$$|a_{i,i}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|, \quad i = 1, 2, \dots, n.$$

   Prove that the Gauss–Seidel method for approximating the the solution of $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is convergent for arbitrary initial values $\boldsymbol{x}_0$.

2. Given a matrix $\mathsf{A} \in \mathbb{R}^{m\times n}$. Find the relation between the singular values of A and the eigenvalues of the matrix

$$\mathsf{B} := \begin{bmatrix} \mathsf{O} & \mathsf{A} \\ \mathsf{A}^T & \mathsf{O} \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}.$$

3. Let $\mathsf{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ with $m \geq n$. We say that $\boldsymbol{z} \in \mathbb{R}^n$ is a least squares solution to "$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$" iff

$$\boldsymbol{z} \in \operatorname{argmin}\{\|\mathsf{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 \mid \boldsymbol{x} \in \mathbb{R}^n\}.$$

   Assume that rank(A)$= n$. Prove that $\boldsymbol{z} \in \mathbb{R}^n$ solves the least squares problem iff it solves the normal equation

$$\mathsf{A}^T\mathsf{A}\boldsymbol{z} = \mathsf{A}^T\boldsymbol{b}.$$

   Argue that the solution $\boldsymbol{z} \in \mathbb{R}^n$ must be unique.

4. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be given. Show that, if $\|\mathsf{A}\| < 1$, for some natural (induced) norm, then $\mathsf{I} - \mathsf{A}$ is non-singular and

$$\frac{1}{1 + \|\mathsf{A}\|} \leq \left\|(\mathsf{I} - \mathsf{A})^{-1}\right\| \leq \frac{1}{1 - \|\mathsf{A}\|}$$

2 Numerical Solutions of Nonlinear Equations

5. Consider the equation $f(x) := x^{n+1} - b^n x + ab^n = 0$, where $n \in \mathbb{N}$ and $a, b \in (0, \infty)$ are given.

   (a) Show that the equation has exactly two distinct positive roots if and only if

$$a < \frac{nb}{(n+1)^{1+\frac{1}{n}}}.$$

   Hint: Analyze the convexity of f.

(b) Assuming the condition established in (a) holds, show that Newton's method converges to the smaller of the two positive roots when started at $x_0 = a$, and to the larger of the two when started at $x_0 = b$.

(c) What happens when $x_0 = \frac{a+b}{2}$?

3 Numerical Solutions of ODEs

6. The BDF2 method for approximating the solution to $u' = f(t, u)$ is given by

$$w^{n+2} - \frac{4}{3}w^{n+1} + \frac{1}{3}w^n = \frac{2}{3}\tau f(t_{n+2}, w^{n+2})$$

Use the boundary locus method to examine the A-stability the method. Is the BDF2 method is A-stable?

7. (a) Find all of the values of α and β so that the 3-step method

$$w^{k+3} + \alpha(w^{k+2} - w^{k+1}) - w^k = \tau\beta\left[f(t_{k+2}, w^{k+2}) + f(t_{k+1}, w^{k+1})\right]$$

is of order 4.

(b) Does the resulting method does satisfy the root condition?

(c) Is the scheme A-stable?

4 Numerical Solutions of PDEs

8. Consider the Lax-Wendroff scheme,

$$w_\ell^{n+1} = w_\ell^n + \frac{a^2\tau^2}{2h^2}\left(w_{\ell-1}^n - 2w_\ell^n + w_{\ell+1}^n\right) - \frac{a\tau}{2h}\left(w_{\ell+1}^n - w_{\ell-1}^n\right),$$

for the approximating the solution of the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, where $a > 0$. Use von Neumann's method to show that the Lax-Wendroff scheme is stable provided the CFL condition

$$\frac{a\tau}{h} \leq 1$$

is enforced.

9. The θ-method for finding approximate solutions to the homogeneous heat equation reads

$$\frac{w^{k+1} - w^k}{\tau} = \Delta_h\left(\theta w^k + (1-\theta)w^{k+1}\right), \quad 0 \leq \theta \leq 1.$$

(a) Find the value or values of θ for which the θ-method is unconditionally stable in $L_\tau^\infty(L_h^\infty)$.

(b) For all other values of θ, state the requirements for conditional stability in $L_\tau^\infty(L_h^\infty)$.

2022 August Numerical Analysis Prelim

1 Numerical Linear Algebra

1. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be Hermitian positive definite (HPD). After k steps of Gaussian elimination without pivoting, A will be reduced to the form

$$\mathsf{A}^{(k)} = \begin{bmatrix} \mathsf{A}_{1,1}^{(k)} & \mathsf{A}_{1,2}^{(k)} \\ \mathsf{O} & \mathsf{A}_{2,2}^{(k)} \end{bmatrix},$$

where $\mathsf{A}_{2,2}^{(k)}$ is an $(n-k)\times(n-k)$ matrix. Prove the following by induction:

   (a) $\mathsf{A}_{2,2}^{(k)}$ is HPD.

   (b) $a_{i,i}^{(k)} \le a_{i,i}^{(k-1)}$, for $k \le i \le n$, $k = 1, 2, \cdots, n-1$.

   (c) Gaussian elimination will continue until completion, and therefore, A has an LU factorization.

2. Suppose that $\mathsf{A} \in \mathbb{R}^{n\times n}$ is SPD. Consider the following Cholesky iteration: .

Step 1: Set $k = 1$ and $\mathsf{A}_0 := \mathsf{A}$.

Step 2: Compute the lower triangular matrix G_k using the Cholesky factorization:

$$\mathsf{A}_{k-1} = \mathsf{G}_k \mathsf{G}_k^T.$$

Step 2: Define

$$\mathsf{A}_k := \mathsf{G}_k^T \mathsf{G}_k.$$

Step 3: Increase k by one. Go back to Step 2.

   (a) Prove that A_k is similar to A_0, for each $k \in \mathbb{N}$ and conclude that the above Cholesky iteration process is well defined by proving that A_k is SPD for each $k \in \mathbb{N}$.

   (b) Now, assume that $n = 2$ and, for concreteness,

$$\mathsf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

   Suppose that A is SPD and $a \ge c$. Applying the Cholesky iteration, show that A_k converges to $\Lambda := \operatorname{diag}(\lambda_1, \lambda_2)$, as $k \to \infty$, where $\lambda_1 \ge \lambda_2 > 0$, and $\lambda_1, \lambda_2 \in \sigma(\mathsf{A})$.

3. Suppose that $m > n$ and the matrix $\mathsf{A} \in \mathbb{C}^{m\times n}$ has the form

$$\mathsf{A} = \begin{bmatrix} \mathsf{A}_1 \\ \mathsf{A}_2 \end{bmatrix},$$

where A_1 is a nonsingular matrix of dimension $n \times n$ and A_2 is an arbitrary matrix of dimension $(m-n) \times n$. Prove that, in general,

$$\left\| \mathsf{A}^{\dagger} \right\|_2 \leq \left\| \mathsf{A}_1^{-1} \right\|_2 ,$$

where $\mathsf{A}^{\dagger} = (\mathsf{A}^H\mathsf{A})^{-1}\mathsf{A}^H$ is the pseudo-inverse.

4. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD) and $\boldsymbol{b} \in \mathbb{R}^n_\star$ be given. To solve the system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, we employ the conjugate gradient (CG) method, starting from the initial guess $\boldsymbol{x}_0 = \boldsymbol{0}$.

   (a) Suppose that A has only two eigenvalues, $0 < \lambda_1 < \lambda_2$. Prove that, for every $\lambda \in (\lambda_1, \lambda_2)$

   $$\|\boldsymbol{e}_1\|_{\mathsf{A}} \leq q(\lambda)\|\boldsymbol{e}_0\|_{\mathsf{A}}, \quad q(\lambda) := \max_{i=1}^{2} \left| 1 - \frac{\lambda_i}{\lambda} \right| .$$

   (b) Prove that

   $$\frac{1}{2}(\lambda_1 + \lambda_2) = \underset{\lambda\in(\lambda_1,\lambda_2)}{\operatorname{argmin}}\, q(\lambda).$$

   Hint: $\max\{|a|, |b|\} = \frac{1}{2}|a+b| + \frac{1}{2}|a-b|$.

   (c) Use the last fact to show that

   $$\|\boldsymbol{e}_1\|_{\mathsf{A}} \leq \frac{\kappa_2 - 1}{\kappa_2 + 1}\|\boldsymbol{e}_0\|_{\mathsf{A}},$$

   where κ_2 is the spectral condition number of A.

2 Numerical Solutions of Nonlinear Equations

5. Consider the equation

$$f(t) := t\ln(t) - c = 0, \tag{1}$$

where $t > 0$, $c \in \mathbb{R}$.

   (a) Under what condition on c does (1) have a *unique* positive root, which we label $t = \alpha$?

   (b) Under the condition of (a), determine the largest interval in which Newton's method, applied to solve (1), converges globally.

   (c) Exhibit the formula for the iterates t_k of Newton's method, and show that, taking $t_0 > \alpha$,

   $$0 < \lim_{k\to\infty} \frac{t_{k+1} - \alpha}{(t_k - \alpha)^2} < \frac{1}{2(1+c)}.$$

3 Numerical Solutions of ODEs

6. For approximating the solution of the IVP

$$\boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0,$$

consider the following multistep method:

$$\sum_{j=0}^{q} a_j \boldsymbol{w}^{k+j} = \tau \sum_{j=0}^{q} b_j \boldsymbol{f}(t_{k+j}, \boldsymbol{w}^{k+j}), \tag{2}$$

where $\{a_j\}_{j=0}^q, \{b_j\}_{j=0}^q \subseteq \mathbb{R}$. For the linear q–step method (2) we define the first and second characteristic polynomials, respectively, as

$$\psi(\mu) := \sum_{j=0}^{q} a_j \mu^j \in \mathbb{P}_q, \qquad \chi(\mu) := \sum_{j=0}^{q} b_j \mu^j \in \mathbb{P}_q.$$

Suppose that

$$\mathrm{Re}[\psi(e^{i\theta})\chi(e^{-i\theta})] = 0. \tag{3}$$

(a) Show that the method is either A-stable or its linear stability domain is empty.

(b) Consider the following one-parameter family of two step methods

$$w^{k+2} - w^k = \tau[\beta f^{k+2} + 2(1-\beta) f^{k+1} + \beta f^k],$$

where $f^k := f(t_k, w^k)$. Show that the scheme satisfies (3). Further, use this fact to show that, if $\beta > \frac{1}{2}$, the method is A-stable.

7. Determine the order of consistency of the following implicit Runge-Kutta scheme: with $\gamma = \sqrt{3}/6$

$$\begin{array}{c|cc} \frac{1}{2}+\gamma & \frac{1}{2}+\gamma & \\ \frac{1}{2}-\gamma & -2\gamma & \frac{1}{2}+\gamma \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

Is the method A-stable?

4 Numerical Solutions of PDEs

8. Consider the advection equation, $u_t + u_x = 0$, and the associated Lax-Friedrichs approximation scheme, given by

$$w_\ell^{n+1} = \frac{1}{2}(1+\mu) w_{\ell-1}^n + \frac{1}{2}(1-\mu) w_{\ell+1}^n, \quad n \in \mathbb{N},$$

where $\mu := \frac{\tau}{h}$.

Numerical Mathematics Preliminary Examination
Tuesday January 18, 2022

I. **Numerical Linear Algebra**

1. (a) Let $\mathsf{A} \in \mathbb{C}^{m\times n}$ be such that there is $\delta \in [0,1]$ for which

$$\left\|\mathsf{A}^H\mathsf{A} - \mathsf{I}\right\|_2 \leq \delta.$$

Show that if σ is a singular value of A, then

$$\sqrt{1-\delta} \leq \sigma \leq \sqrt{1+\sigma}.$$

(b) A matrix $\mathsf{A} \in \mathbb{C}^{n\times n}$ is said to satify the Cordes condition if there is $\varepsilon \in (0,1]$ for which

$$\frac{\|\mathsf{A}\|_F^2}{(\operatorname{tr}\mathsf{A})^2} \leq \frac{1}{n-1+\varepsilon}.$$

Show that, if $n=2$, every SPD matrix satisfies the Cordes condition.

2. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be nonsingular and $\boldsymbol{f} \in \mathbb{C}^n$. Denote by $\boldsymbol{x} \in \mathbb{C}^n$ the (unique) solution of

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{f}.$$

To approximate $\boldsymbol{x}$, starting from an arbitrary $\boldsymbol{x}_0 \in \mathbb{C}^n$, we use the following iterative scheme

$$\boldsymbol{x}_{k+1} = (\mathsf{I} + \mathsf{BA})\boldsymbol{x}_k - \mathsf{B}\boldsymbol{f},$$

where $\mathsf{B} \in \mathbb{C}^{n\times n}$ is some nonsingular matrix.

(a) Show that if there is $\rho < 1$ for which

$$\sigma(\mathsf{BA}) \subset \{z \in \mathbb{C} \mid |z+1| < \rho\},$$

then this method converges.

(b) Let A be HPD and strictly diagonally dominant, and set $\mathsf{B} = c\mathsf{I}$, where $c < 0$ satisfies

$$|c| < \frac{2}{\max_i \sum_{k=1}^n |a_{i,k}|}.$$

Show that the method converges.
Hint: Use the previous step and Gershgorin circle theorem to estimate the eigenvalues of the matrix BA.

3. Let $\mathsf{U} \in \mathbb{R}^{n\times n}$ be orthogonal. Show that it can be written as the product of at most n nontrivial (that is, non-identity) Householder reflections.

II. **Numerical Solutions of Nonlinear Equations**

4. Let $f : \mathbb{R} \to \mathbb{R}$ be sufficiently smooth and such that there is a unique $\xi \in \mathbb{R}$ for which

$$f(\xi) = 0, \qquad f'(\xi) \neq 0.$$

To approximate ξ we consider the two-step Newton method

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad x_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)}$$

Show that

(a) If the method converges, then

$$\lim_{k\to\infty} \frac{x_{k+1} - \xi}{(y_k - \xi)(x_k - \xi)} = \frac{f''(\xi)}{f'(\xi)}.$$

(b) The convergence is cubic and

$$\lim_{k\to\infty} \frac{x_{k+1} - \xi}{(x_k - \xi)^3} = \frac{1}{2}\left[\frac{f''(\xi)}{f'(\xi)}\right]^2.$$

III. Numerical Solutions of ODEs

5. Recall that all Runge-Kutta (RK) schemes are uniquely described by their Butcher tableau

$$\begin{array}{c|c} \mathbf{c} & \mathsf{A} \\ \hline & \mathbf{b}^{\mathsf{T}} \end{array}.$$

A *diagonally implicit* RK (DIRK) scheme is one where A is lower triangular.
Derive all the second order, two-stage, DIRK schemes, such that

(a) The A matrix has positive diagonal entries:

$$\mathbf{e}_i^{\mathsf{T}} \mathsf{A} \mathbf{e}_i > 0, \qquad i = 1, 2,$$

and $\{\mathbf{e}_i\}_{i=1}^2$ is the canonical basis of $\mathbb{R}^2$.

(b) The second stage coincides with the solution at the next step:

$$\mathsf{A}^{\mathsf{T}} \mathbf{e}_2 = \mathbf{b}.$$

Show that if, in addition, we wish that the diagonal entries coincide:

$$\mathbf{e}_1^{\mathsf{T}} \mathsf{A} \mathbf{e}_1 = \mathbf{e}_2^{\mathsf{T}} \mathsf{A} \mathbf{e}_2$$

then the only possible scheme is the so-called *Alexander scheme*:

$$\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array} \qquad \gamma = 1 - \frac{1}{\sqrt{2}}. \tag{1}$$

6. To approximate the solution of the initial value problem

$$u'(t) = f(t, u), \quad t > 0; \qquad u(0) = u_0,$$

we apply the multistep scheme

$$w^{k+2} = w^{k-1} + \frac{3}{8}\tau \left[f(t_{k+2}, w^{k+2}) + 3f(t_{k+1}, w^{k+1}) + 3f(t_k, w^k) + f(t_{k-1}, w^{k-1}) \right],$$

where, as usual, $\tau > 0$ is the (constant) timestep and $t_k = k\tau$. Study the consistency and stability of this scheme.

7. Investigate the A-stability of Alexander scheme (1).

IV. **Numerical Solutions of PDEs**

8. Let $N \in \mathbb{N}$. Define the meshsize $h = 1/N > 0$, the grid

$$\bar{\Omega}_h = \{jh\}_{j=0}^N,$$

and the space of grid functions $w_h : \bar{\Omega}_h \to \mathbb{R}$ as $\mathcal{V}(\bar{\Omega}_h)$. Define also

$$\mathcal{V}_0(\bar{\Omega}_h) = \left\{ w_h \in \mathcal{V}(\bar{\Omega}_h) \,\middle|\, w_h(0) = w_h(1) = 0 \right\}.$$

For $p \in (1, \infty)$ the L_h^p–norm on $\mathcal{V}_0(\bar{\Omega}_h)$ is

$$\|w_h\|_{L_h^p} = \left(h \sum_{j=1}^{N-1} |w_h(jh)|^p \right)^{1/p}.$$

Show that, if $p < q$, we have:

(a) Discrete embedding

$$\|w_h\|_{L_h^p} \leq \|w_h\|_{L_h^q}.$$

(b) Interpolation

$$\|w_h\|_{L_h^q} \leq \|w_h\|_{L_h^p}^{p/q} \|w_h\|_{L_h^\infty}^{1-p/q}.$$

9. Let $L > 0$, $a > 0$, and $c \geq 0$. Assume that $f : (0, L) \to \mathbb{R}$ is given and sufficiently smooth. Write a weak formulation of the problem

$$-au''(x) + cu(x) = f(x), \; x \in (0, L), \qquad u(0) = u(L) = 0.$$

Find an approximate solution to this problem using Galerkin's method over the space

$$W = \operatorname{span} \left\{ \varphi_k(x) = \sin\left(\frac{\pi k x}{L} \right) \right\}_{k=1}^{20}.$$

10. Let $\Omega = (0,1)$, $T > 0$, and $u_0 : \Omega \to \mathbb{R}$ be given. To approximate the solution of the heat equation

$$\begin{cases} \partial_t u(x,t) - \partial^2_{xx} u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \end{cases}$$

via finite differences, we introduce a space-time grid with space step $h > 0$ and time step $\tau > 0$, and consider the so-called *Richardson* method: $w^0 = w^1 = \pi_h u_0$ and

$$\frac{1}{2\tau}\left(w^{k+1} - w^{k-1}\right) - \Delta_h w^k = 0, \quad k = 1, \dots, K.$$

Show that:

(a) The consistency error of the scheme is $\mathcal{O}(\tau^2 + h^2)$.

(b) The method is unconditionally unstable in the $L^\infty_\tau(L^2_h)$ norm.
Hint: Recall that
$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

Numerical Mathematics Preliminary Examination
Wednesday August 11, 2021

I. Numerical Linear Algebra

1. The purpose of this problem is to use the the spectral decomposition theorem to prove the existence of the SVD. For this reason, you cannot invoke the SVD to solve it. Let $\mathsf{A} \in \mathbb{C}^{m\times n}$, with $m \geq n$.

   (a) Show that both AA^H and $\mathsf{A}^H\mathsf{A}$ are Hermitian and positive semi-definite. Therefore, they admit the decompositions

   $$\mathsf{AA}^H = \mathsf{USS}^T\mathsf{U}^H, \quad \mathsf{A}^H\mathsf{A} = \mathsf{VS}^T\mathsf{SV}^H,$$

   where $\mathsf{U} \in \mathbb{C}^{m\times m}$ and $\mathsf{V} \in \mathbb{C}^{n\times n}$ are unitary, and $\mathsf{S} \in \mathbb{R}^{m\times n}$ is diagonal and has non-negative entries. Notice that this same S must appear in both decompositions.

   (b) From the previous two decompositions, show that

   $$\mathsf{A} = \mathsf{USV}^H.$$

2. Suppose that $\mathsf{A} \in \mathbb{C}^{n\times n}$ is Hermitian with spectrum $\sigma(\mathsf{A}) = \{\lambda_i\}_{i=1}^n \subset \mathbb{R}$ and the associated orthonormal basis of eigenvectors $S = \{\boldsymbol{w}_i\}_{i=1}^n$. Given $\varepsilon > 0$, suppose that $\boldsymbol{x} \in \mathbb{C}_\star^n$ is a unit vector ($\|\boldsymbol{x}\|_2 = 1$) satisfying

   $$0 < \|\boldsymbol{x} - \boldsymbol{w}_k\|_2 < \varepsilon,$$

   for some $k \in \{1, \ldots, n\}$. Prove that

   $$\left|\boldsymbol{x}^H\mathsf{A}\boldsymbol{x} - \lambda_k\right| < 2\rho(\mathsf{A})\varepsilon^2,$$

   where $\rho(\mathsf{A})$ is the spectral radius of A.
   Hint: If you can not prove this result, prove a simpler one.

3. Let $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{n\times n}$ be invertible and $\boldsymbol{b} \in \mathbb{C}^n$. Prove that, if A is strictly diagonally dominant, *i.e.*,

   $$|a_{i,i}| > \sum_{k\neq i} |a_{i,k}| \ , \quad \forall\ i = 1, \ldots, n,$$

   then for any starting value $\boldsymbol{x}_0$, the classical Jacobi iteration method for approximating the solution to $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is convergent.

4. Let $\mathsf{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{C}^m$, with $m > n$. Assume that $\operatorname{rank}(A) = n$, and let $\mathsf{A} = \hat{\mathsf{Q}}\hat{\mathsf{R}}$ be a reduced QR factorization of A and $\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^H$ be a singular value

decomposition (SVD) of A. Recall that the *Moore-Penrose pseudoinverse* of A is defined by

$$\mathsf{A}^{\dagger} = \mathsf{V}\Sigma^{\dagger}\mathsf{U}^{H},$$

where $\Sigma^{\dagger} = \mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0) \in \mathbb{R}^{n\times m}$ and $r \le n$ is the rank of A. Show that $\boldsymbol{x} \in \mathbb{C}^n$ is a least squares solution to $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ iff $\hat{\mathsf{R}}\boldsymbol{x} = \hat{\mathsf{Q}}^{H}\boldsymbol{b}$ iff $\boldsymbol{x} = \mathsf{A}^{\dagger}\boldsymbol{b}$.

II. Numerical Solutions of Nonlinear Equations

5. Suppose that $f \in C^1([a,b];\mathbb{R})$, and, for some $\xi \in (a,b)$, $f(\xi) = 0$. Assume that there are positive constants $m, M \in \mathbb{R}$, such that $0 < m \le f'(x) \le M$, for all $x \in [a,b]$. To approximate the zero ξ, consider the following algorithm: given $x_0 \in [a,b]$, compute $x_1, x_2, \ldots,$ via

$$\tau^{-1}(x_{k+1} - x_k) + f(x_k) = 0, \qquad k = 0, 1, 2, \ldots,$$

where $\tau \neq 0$ is a parameter to be determined.

(a) Prove that there is one and only one zero in $[a,b]$.

(b) Show that if $0 < \tau < 2/M$ the method converges, provided x_0 is sufficiently close to ξ.

(c) Show that the optimal value of τ is given by $\tau_0 = \frac{2}{m+M}$ and that, in this case, we have the error estimate:

$$|\xi - x_k| \le q^k |\xi - x_0|, \qquad q = \frac{M-m}{M+m}.$$

III. Numerical Solutions of ODEs

6. Consider

$$u'(t) = f\left(t, u(t)\right), \qquad t \ge 0, \qquad u(0) = u_0,$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous in its first variable and Lipschitz continuous in its second variable.

(a) Prove that the forward Euler method converges.

(b) Prove that the backward Euler method converges.

Clearly indicate the smoothness assumptions you are using to prove convergence. You may assume the appropriate local truncation error estimates without proof.

7. Show that the BDF3 method

$$w^{k+3} - \frac{18}{11}w^{k+2} + \frac{9}{11}w^{k+1} - \frac{2}{11}w^{k} = \frac{6}{11}sf(t_{k+3}, w^{k+3})$$

satisfies the root condition and is of order 3. Conclude, therefore, that it must be a convergent method. Use the boundary locus method to prove that BDF3 cannot be A-stable.

IV. **Numerical Solutions of PDEs**

8. A general linear explicit finite difference method for approximating a time-dependent Cauchy problem can be written in the form

$$w_j^{n+1} = \sum_{p \in P} a_p w_{j+p}^n,$$

where P is a finite subset of $\mathbb{Z}$ and a_p are the weights, which depend upon the time and space step sizes. We say that the method *reproduces the constant state* if, whenever $w^n \equiv 1$, we obtain that $w^{n+1} \equiv 1$.

   (a) Show that if the method reproduces the constant state, then

   $$\sum_{p \in P} a_p = 1.$$

   (b) A method is *max-norm non-increasing* iff $\|w^{n+1}\|_{L_h^\infty} \leq \|w^n\|_{L_h^\infty}$. A method is *positivity preserving* iff whenever $w_\ell^n \geq 0$, for all $\ell \in \mathbb{Z}$, then $w_\ell^{n+1} \geq 0$, for all $\ell \in \mathbb{Z}$. Show that if a method reproduces the constant state and is max-norm non-increasing, then the method is positivity preserving.
   Hint: Suppose that $w_\ell^n \geq 0$ and $w_k^n = \|w^n\|_{L_h^\infty} = \alpha \geq 0$, for some $k \in \mathbb{Z}$. Now define a new variable $\eta_\ell^n := w_\ell^n - \alpha/2$. Apply the method to η^n, and use the fact that $-\alpha/2 \leq \eta_\ell^{n+1} \leq \alpha/2$, for all $\ell \in \mathbb{Z}$, to conclude the result.

   (c) Show that, if a method reproduces the constant state and is max-norm non-increasing, then $a_p \geq 0$ for all $p \in P$.

9. Consider the diffusion problem

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1, \quad 0 < t \leq T \\ u(0,t) &= \phi_0(t), & 0 < t \leq T, \\ u(1,t) &= \phi_1(t), & 0 < t \leq T, \\ u(x,0) &= g(x), & 0 \leq x \leq, \end{aligned}$$

where $g(0) = \phi_0(0)$ and $g(1) = \phi_1(0)$ for consistency. Define $h = \frac{1}{m+1}$, $\tau = \frac{T}{N}$, $\mu = \frac{\tau}{h^2}$, $x_\ell = \ell \cdot h$, $t_n = n \cdot \tau$, $g_\ell = g(x_\ell)$, $\phi_i^n = \phi_i(t_n)$, $i = 0, 1$. The forward Euler method for this problem is define as follows:

$$w_\ell^{n+1} = w_\ell^n. + \mu \left(w_{\ell-1}^n - 2w_\ell^n + w_{\ell+1}^n \right), \quad 1 \leq \ell \leq m, \quad 0 \leq n \leq N-1,$$

with $w_0^n = \phi_0^n$, $w_{m+1}^n = \phi_1^n$, $0 \leq n \leq N$, and $w_\ell^0 = g_\ell, 1 \leq \ell \leq m$. Suppose $u \in C^4([0,1] \times [0,T])$ and define $u_\ell^n = u(x_\ell, t_n)$. Define $e_\ell^n := u_\ell^n - w_\ell^n$, $0 \leq \ell \leq m+1$, $0 \leq n \leq N$, $\boldsymbol{e}^n = [e_1^n, \ldots, e_m^n]^T$, $0 \leq n \leq N$. Suppose that $\mu = \mu_0 \leq \frac{1}{2}$. Prove that

$$\max_{0 \leq n \leq N} \|\boldsymbol{e}^n\|_\infty \leq C h^2,$$

where $C > 0$ is independent of h and τ. You may assume the appropriate estimate of the local truncation error with proof.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION: AUGUST 2020

Instructions

Read each problem carefully. You must show your work to receive credit. If you believe a problem has a typo, missing conditions, or can be interpreted in several ways, please clearly indicate so in your work. In this case, fix the problem in a way that it does not become trivial.

Please stay safe, wear a mask, and be mindful of social distancing.

1. Numerical Linear Algebra

1. The purpose of this problem is to provide a proof of existence of the SVD. For this reason, you *cannot use* the SVD to solve it.

   Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$.

   a) Show that both AA^* and A^*A are Hermitian and positive semidefinite. Therefore they admit the decompositions
   $$AA^* = U\Sigma\Sigma^\intercal U^*, \qquad A^*A = V\Sigma^\intercal \Sigma V^*,$$
   where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ is diagonal and has nonnegative entries. Notice that this must be the same matrix in both decompositions.

   b) From the previous two decompositions show that $A = U\Sigma V^*$. *Hint:* Show that the columns of U are eigenvectors of AA^*.

2. Let $A \in \mathbb{C}^{n\times n}$. Recall that we say that A is strictly diagonally dominant of dominance $\delta > 0$ if
   $$|a_{i,i}| \geq \delta + \sum_{j\neq i} |a_{i,j}|, \quad i = 1, \dots, n.$$
   Let A be strictly diagonally dominant of dominance $\delta > 0$.

   a) Show that A is nonsingular and that
   $$\|A^{-1}\|_\infty \leq \delta^{-1}.$$

   b) Show that if we apply Gaussian elimination without pivoting to
   $$Ax = f,$$
   then the procedure reaches completion without encountering any zero pivot elements.

   c) Assume, in addition, that $A \in \mathbb{R}^{n\times n}$. Show that all the entries of A^{-1} are nonnegative.

3. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ be full rank, and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization of A.

   a) Show that $P = \hat{Q}\hat{Q}^*$ is an orthogonal projection onto $\mathcal{R}(A)$, the range of A.

   b) Let $f \in \mathbb{C}^m$. Show that the vector $x \in \mathbb{C}^n$ is a least squares solution to $Ax = f$ iff $Ax = Pf$, where P is the orthogonal projection onto $\mathcal{R}(A)$.

4. Let $A \in \mathbb{R}^{n\times n}$ be an SPD matrix with $\sigma(A) = \{\lambda_i\}_{i=1}^n$ and $0 < \lambda_1 \le \ldots \le \lambda_n$. Denote by $\mathbb{P}_k$ the space of polynomials of degree at most $k \in \mathbb{N}$, that take the value 1 at zero, and

$$\tilde{\mathbb{P}}_k = \{p \in \mathbb{P}_k : p(\lambda_n) = 0\}.$$

a) Show that the error in CG satisfies

$$\|x - x_k\|_A \le \|x - x_0\|_A \inf_{p\in\tilde{\mathbb{P}}_k} \max_{\tau\in[\lambda_1,\lambda_{n-1}]} |p(\tau)|.$$

b) From the previous item show that the error in CG satisfies

$$\|x - x_k\|_A \le \|x - x_0\|_A \frac{\lambda_n - \lambda_1}{\lambda_n} \inf_{p\in\mathbb{P}_{k-1}} \max_{\tau\in[\lambda_1,\lambda_{n-1}]} |p(\tau)|.$$

2. Numerical Solution of Nonlinear Equations

1. Consider the nonlinear equation $e^x = \sin x$.
   a) Show that there is a solution $x_* \in (-5/4\pi, -\pi)$.
   b) Consider the following iterative methods:

$$x_{k+1} = \ln\sin(x_k), \qquad x_{k+1} = \arcsin e^{x_k}.$$

   What can you say about the local convergence of these methods? About their convergence order?
   c) Provide a method that converges quadratically to x_*. You may invoke a Theorem, but you must precisely state it and verify its conditions.

3. Numerical Solution of ODEs

1. Consider the RK methods given by the tables

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \qquad \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

   a) Show that these methods satisfy all necessary order conditions to be of order at least two.
   b) Show one of these methods is a collocation method, while the other one is not. For the collocation method find its order of consistency.
2. Show that the explicit multistep method

$$w^{k+3} + \alpha_2 w^{k+2} + \alpha_1 w^{k+1} + \alpha_0 w^k = s\big[\beta_2 f(t_{k+2}, w^{k+2}) + \beta_1 f(t_{k+1}, w^{k+1}) + \beta_0 f(t_k, w^k)\big]$$

   for approximating the solution to the initial value problem

$$u'(t) = f(t, u(t)), \quad u(0) = u_0$$

   is fourth order only if $\alpha_0 + \alpha_2 = 8$ and $\alpha_1 = -9$. Prove that this method cannot be both fourth order and convergent.

4. Numerical Solution of PDEs

1. Let $\mathcal{G}_h$ be a uniform grid, of spacing h, of the unit square $(0,1)^2$ and let $\mathbb{V}_h$ denote the space of grid functions that vanish on the boundary of $(0,1)^2$. Recall that the discrete (finite difference) Laplacian is defined on $\mathbb{V}_h$ via
$$(\Delta_h w)_{i,j} = \frac{w_{i-1,j} + w_{i+1,j} + w_{i,j-1} + w_{i,j+1} - 4w_{i,j}}{h^2},$$
where $(ih, jh) \in \mathcal{G}_h$ are in the interior.
   We say that a function $w \in \mathbb{V}_h$ is *discrete subharmonic* if
$$\Delta_h w \geq 0.$$
Show that a discrete subharmonic function attains its maximum at the boundary.
2. Let $a, b \in \mathbb{R}$, $b - a = \ell > 0$. Given a function $w \in C([a,b])$ define $W \in \mathbb{P}_1$ by:
$$W(a) = w(a), \qquad \int_a^b W(x)dx = \int_a^b w(x)dx.$$
show that there is a constant, independent of ℓ, such that for every w with $w'' \in L^2(a,b)$ we have
$$\|w - W\|_{L^2(a,b)} \leq C\ell^2 \|w''\|_{L^2(a,b)}.$$
Hint: Recall that, by passing through the Taylor polynomial, one must only check two conditions.
3. Find the values of θ for which the θ-method for the discretization of the heat equation on a bounded interval is unconditionally stable in $L^\infty_s(L^2_h)$.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION: AUGUST 2019

1. Numerical Linear Algebra

1. Let $\mathsf{U} \in \mathbb{C}^{n \times n}$ be unitary, *i.e.*, $\mathsf{U}^H = \mathsf{U}^{-1}$, where U^H denotes the conjugate transpose of U. Recall that a Householder reflector is a matrix of the form

$$\mathsf{H} = \mathsf{I}_n - 2\frac{\boldsymbol{v}\boldsymbol{v}^H}{\boldsymbol{v}^H\boldsymbol{v}},$$

where $\boldsymbol{v} \in \mathbb{C}^n_\star := \mathbb{C}^n \setminus \{\mathbf{0}\}$. Prove that

$$\mathsf{U} = \mathsf{H}_1 \cdots \mathsf{H}_\ell \mathsf{D},$$

for some $\ell \in \{1, \cdots, n-1\}$, where H_i is a Householder matrix for each $i \in \{1, \cdots, \ell\}$, and D is a diagonal matrix whose diagonal entries each have modulus 1. (If you cannot prove this, perhaps you can prove the following, slightly simpler result: If U is real and orthogonal, then it is the product of at most n (real) Householder matrices.)

2. Let $\mathsf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ with $m \geq n$. We say that $\boldsymbol{z} \in \mathbb{R}^n$ is a least squares solution to "$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$" iff

$$\boldsymbol{z} \in \operatorname{argmin}\left\{\|\mathsf{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 \;\middle|\; \mathbf{x} \in \mathbb{R}^n\right\}.$$

Assume that $\operatorname{rank}(\mathsf{A}) = n$. Prove that $\boldsymbol{z} \in \mathbb{R}^n$ solves the least squares problem iff it solves the normal equation

$$\mathsf{A}^T\mathsf{A}\boldsymbol{z} = \mathsf{A}^T\boldsymbol{b}.$$

Argue that the solution $\boldsymbol{z} \in \mathbb{R}^n$ must be unique.

3. Suppose $\mathsf{A} \in \mathbb{C}^{n \times n}$ is Hermitian and that the spectrum of A is denoted $\sigma(\mathsf{A}) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$. Let $S = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ be an orthonormal basis of eigenvectors of A, with $\mathsf{A}\boldsymbol{w}_k = \lambda_k \boldsymbol{w}_k$, for $k = 1, \ldots, n$. The Rayleigh quotient of $\boldsymbol{x} \in \mathbb{C}^n_\star$ is defined as

$$R(\boldsymbol{x}) := \frac{\boldsymbol{x}^H\mathsf{A}\boldsymbol{x}}{\boldsymbol{x}^H\boldsymbol{x}}.$$

Suppose that for some k, $\|\boldsymbol{w} - \boldsymbol{w}_k\|_2 \leq \epsilon$, where $\|\boldsymbol{w}\|_2 = 1$. Show that

$$|R(\boldsymbol{w}) - \lambda_k| \leq 2\rho(\mathsf{A})\epsilon^2.$$

(If you cannot prove this, perhaps you can show the somewhat simpler result: $\|\boldsymbol{w} - \boldsymbol{w}_k\|_2 = O(\epsilon)$ and $\|\boldsymbol{w}\|_2 = 1$ implies $|R(\boldsymbol{w}) - \lambda_k| = O(\epsilon^2)$.)

4. Let $\mathsf{A} \in \mathbb{C}^{n \times n}$ be Hermitian positive definite (HPD), represented as

$$\mathsf{A} = \begin{bmatrix} \alpha & \boldsymbol{p}^H \\ \boldsymbol{p} & \hat{\mathsf{A}} \end{bmatrix},$$

where α is a scalar, $\boldsymbol{p} \in \mathbb{C}^{n-1}$, and $\hat{\mathsf{A}} \in \mathbb{C}^{(n-1)\times(n-1)}$. After 1 step of Gaussian elimination (without pivoting), A will be reduced to the matrix

$$\begin{bmatrix} \alpha & \boldsymbol{p}^H \\ \mathbf{0} & \mathsf{B} \end{bmatrix},$$

where $\mathsf{B} \in \mathbb{C}^{(n-1)\times(n-1)}$. Prove that B is HPD. In so doing, also prove that the corresponding diagonal elements of B are smaller than those of $\hat{\mathsf{A}}$.

2. Numerical Solution of Nonlinear Equations

1. Suppose that $f : [a, b] \to \mathbb{R}$ and, for some $\xi \in [a, b]$, $f(\xi) = 0$. Assume further that $f \in C^2[a, b]$, and f' and f'' are strictly positive on the interval $[a, b]$. Prove that, for any starting value $x_0 \in (\xi, b]$, the sequence $\{x_k\}$ defined by Newton's method converges quadratically to the root ξ as $k \to \infty$.

3. Numerical Solution of ODEs

1. Find all of the values of α and β so that the 3-step method

$$w^{k+3} + \alpha(w^{k+2} - w^{k+1}) - w^k = s\beta\left[f(t_{k+2}, w^{k+2}) + f(t_{k+1}, w^{k+1})\right]$$

is of order 4. Show that the resulting method does not satisfy the root condition and is, therefore, not convergent. Is the scheme A-stable? (Why or why not?)

4. Numerical Solution of PDEs

1. Suppose that $u : [0, 1] \times [0, T] \to \mathbb{R}$ is a solution to the diffusion problem,

$$\begin{aligned} \partial_t u &= \partial_{xx} u, && 0 < x < 1,\ 0 < t \le T \\ u(0, t) &= \phi_0(t), && 0 \le t \le T, \\ u(1, t) &= \phi_1(t), && 0 \le t \le T, \\ u(x, 0) &= g(x), && 0 \le x \le 1. \end{aligned}$$

Let $m, N \in \mathbb{N}$ be given, and define $h := \frac{1}{m+1}$, $s := \frac{T}{N}$, $\mu := \frac{s}{h^2}$, $x_\ell = \ell \cdot h$, $t_n = n \cdot s$. The backward Euler scheme for the diffusion problem is defined as follows:

$$w_\ell^{n+1} = w_\ell^n + \mu\left(w_{\ell-1}^{n+1} - 2w_\ell^{n+1} + w_{\ell+1}^{n+1}\right), \quad 1 \le \ell \le m, \quad 0 \le n \le N-1,$$

with $w_0^n = \phi_0^n$, $w_{m+1}^n = \phi_1^n$, $0 \le n \le N$, and $w_\ell^0 = g_\ell$, $1 \le \ell \le m$. Suppose $u \in C^4([0, 1] \times [0, T])$ and define $u_\ell^n = u(x_\ell, t_n)$ and $e_\ell^n := u_\ell^n - w_\ell^n$, $0 \le \ell \le m+1$, $0 \le n \le N$, $e^n = [e_1^n, \ldots, e_m^n]^T$, $0 \le n \le N$. Prove that

$$\max_{0 \le n \le N} \|e^n\|_\infty \le C(h^2 + s),$$

where $C > 0$ is independent of h and s. (You may assume the appropriate form of the local truncation error, that is, the order of the method, without proof.)

2. The Lax-Friedrichs scheme for the linear advection equation $\partial_t u + a\partial_x u = 0$ may be expressed as

$$w_\ell^{n+1} = \frac{1}{2}(1+\mu)w_{\ell-1}^n + \frac{1}{2}(1-\mu)w_{\ell+1}^n,$$

where $\mu = \frac{as}{h}$, $a > 0$, $s > 0$ is the time step size, and $h > 0$ is the space step size. Use the von Neumann stability analysis to show that the scheme is stable provided the CFL condition

$$0 < \mu = \frac{as}{h} \leq 1$$

holds.

3. Suppose that $u \in C^4([0,1])$ is the solution to the one dimensional Poisson problem

$$-\frac{d^2u}{dx^2} = f \quad \text{in} \quad \Omega = (0,1), \qquad u(0) = g_0; \ u(1) = g_1,$$

and w is the finite difference approximation satisfying

$$-\Delta_h w_i = f(p_i) =: f_i, \ i \in \{1, \cdots, m\}, \quad w_0 = g_0; \ w_{m+1} = g_1,$$

where $h = \frac{1}{m+1}$, $p_i = i \cdot h$. Define $e_i := u(p_i) - w_i$, $i \in \{0, \cdots, m+1\}$. Prove that

$$\|e\|_{2,h} \leq Ch^2,$$

where $C > 0$ is independent of h and

$$\|e\|_{2,h} := \sqrt{h\sum_{i=1}^{m} |e_i|^2}.$$

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION: AUGUST 2018

1. Numerical Linear Algebra

1. Let $A \in \mathbb{C}^{n\times n}$ have an SVD $A = U\Sigma V^\star$. Find an eigenvalue decomposition of the matrix
$$\begin{pmatrix} 0 & A^\star \\ A & 0 \end{pmatrix}.$$
2. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Suppose that $\lambda \in \sigma(A)$ has multiplicity one and
$$\inf\{|\lambda - \mu| : \mu \in \sigma(A) \setminus \{\lambda\}\} = d > 0.$$
   a) Let $\theta \in \mathbb{R}$ be such that
$$0 < |\lambda - \theta| \le \frac{d}{2}.$$
   Show that $(A - \theta I)^{-1}$ exists and that, if $\mu \in \sigma(A)$, then $(\mu - \theta)^{-1} \in \sigma((A - \theta I)^{-1})$ with the same eigenvector.
   b) Let $w_\lambda \in \mathbb{R}^n$ be the eigenvector of A associated with λ, and let $v_0 \in \mathbb{R}^n$ be such that $\|v_0\|_{\ell^2} = 1$, and $(v_0, w_\lambda)_{\ell^2} \neq 0$. Consider the following *inverse power iteration scheme*:
$$\zeta_{k+1} = (A - \theta I)^{-1} v_k, \qquad v_{k+1} = \frac{\zeta_{k+1}}{\|\zeta_{k+1}\|_{\ell^2}}.$$
   Show that there is a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \in \{-1, 1\}$, such that $\varepsilon_k v_k \to w_\lambda$ as $k \to \infty$.
3. Let V be a vector space with scalar product $(\cdot,\cdot)$, and let $A, B \in \mathcal{L}(V)$ be positive and self–adjoint with respect to the given scalar product (think of $V = \mathbb{C}^n$ and A, B being HPD matrices).
   a) Introduce the so–called *energy norm* of B:
$$\|x\|_B^2 = (Bx, x), \quad \forall x \in V.$$
   Show that this is indeed a norm. Show that this norm comes from an inner product. Call this inner product $(\cdot,\cdot)_B$.
   b) For $C \in \mathcal{L}(V)$, let $\kappa_B(C)$ be the condition number of C in the energy norm of B. Assume that there are constants $\gamma_1, \gamma_2 > 0$ such that
$$\gamma_1 (Bx, x) \le (Ax, x) \le \gamma_2 (Bx, x), \quad \forall x \in V.$$
   Show that
$$\kappa_B(B^{-1}A) \le \frac{\gamma_2}{\gamma_1}.$$
4. Assume we approximate the solution to a linear system of equations via the linear, stationary scheme

$$x_{k+1} = Tx_k + c. \tag{1}$$

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In this problem we will study an acceleration procedure known as *extrapolation.* The scheme (1) can be embedded in a one–parameter family of methods:

$$x_{k+1} = \gamma(Tx_k + c) + (1-\gamma)x_k = T_\gamma x_k + \gamma c, \tag{2}$$

with $T_\gamma \quad \gamma T + (1-\gamma)I$, for some $\gamma > 0$.

a) Show that any fixed point of (2) is a fixed point of (1).

b) Assume that we know that $\sigma(T) \subset [a, b]$. Show that $\sigma(T_\gamma \;\subset [\gamma a + 1 - \gamma, \gamma b + 1 - \gamma]$.

c) Show that

$$\rho(T_\gamma \;\leq \max_{a \leq \lambda \leq b} |\gamma\lambda + 1 - \gamma| = \varrho_\gamma.$$

d) Show that if $1 \notin [a, b]$ then the choice $\gamma^\star = 2/(2 - a - b)$ minimizes ϱ_γ and that, in this case,

$$\varrho_{\gamma^\star} = 1 - |\gamma^\star|d,$$

where d is the distance between 1 and $[a, b]$. This shows that, using extrapolation, we can turn a non–convergent scheme into a convergent one.

e) Consider Richardson method with $\alpha = 1$ and $A = A^\star > 0$. If $\sigma(A) \subset [m, M]$, show that choosing $\gamma = 2/(m + M)$ we obtain

$$\varrho_\gamma = \frac{M - m}{M + m},$$

thus Richardson method with extrapolation converges.

2. Numerical Solution of Nonlinear Equations

1. The purpose of this problem is to construct root finding iterative schemes that have order $p = 2$ and $p = 3$. Assume that $f : \mathbb{R} \to \mathbb{R}$ has a simple root $x \in \mathbb{R}$, that is $f(x) = 0$. Assume also that $f(x) = 0$ can be rewritten as $x = \varphi(x)$, for some φ. We consider the iterative scheme:

$$x_{k+1} = \varphi(x_k).$$

a) Define the error $e_k = x_k - x$. Show that

$$e_k = \varphi'(x)e_{k-1} + \mathcal{O}(|e_{k-1}|^2).$$

b) Assume that the method converges. Show that if $\varphi'(x) \neq 0$ and φ'' is bounded, then the method converges linearly.

c) Assume that the method converges. Show that if $\varphi'(x) = 0$ but $\varphi''(x) \neq 0$ the method converges quadratically.

d) Define

$$\psi(x) = x + a_1(x)f(x) + a_2(x)[f(x)]^2.$$

Find conditions so that $x = \psi(x)$ iff $f(x) = 0$.

e) Evaluate the first and second derivative of ψ w.r.t. x. From them show that if $p = 2$ (the method converges quadratically) then we obtain Newton's method.

f) Show that if $p = 3$

$$a_1(x) = -\frac{1}{f'(x)}, \qquad a_2(x) = -\frac{f''(x)}{2[f'(x)]^3}.$$

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3. Numerical Solution of ODEs

1. Consider the two-step method

$$y_{n+2} - y_n = 2hf(t_{n+1}, y_{n+1}).$$

This is usually called the *explicit midpoint rule*.
a) Study the order, stability and convergence of this method.
b) Find the linear stability domain of this method.

4. Numerical Solution of PDEs

1. Let $\Omega = (0,1)$ and $\bar{\Omega}_h$ be a uniform mesh on Ω with size h. Let $\mathring{\mathcal{U}}_h$ be the space of mesh functions such that $w_h(0) = w_h(1) = 0$. Denote by δ and $\bar{\delta}$, respectively, the forward and backward difference operators on $\mathring{\mathcal{U}}_h$.
   a) Show the following *discrete embedding*: There is a constant $C > 0$ independent of h such that, for all w_h in $\mathring{\mathcal{U}}_h$, we have
   $$\|w_h\|_{L^\infty_h(\Omega_h)} \le C\|\bar{\delta} w_h\|_{L^2_h(\Omega_h)}.$$
   b) Show the following *discrete Poincaré inequality*: There is $C > 0$ independent of h such that, for all $w_h \in \mathring{\mathcal{U}}_h$
   $$\|w_h\|_{L^2(\Omega_h)} \le C\|\bar{\delta} w_h\|_{L^2_h(\Omega_h)}.$$
   c) Show the following *summation by parts formula*:
   $$-(\delta v_h, w_h)_{L^2_h(\Omega_h)} = (v_h, \bar{\delta} w_h)_{L^2_h(\Omega_h)}, \quad \forall v_h, w_h \in \mathring{\mathcal{U}}_h.$$
   d) Consider the problem: Find $u_h \in \mathring{\mathcal{U}}_h$ such that
   $$-\delta\bar{\delta} u_h = f_h.$$
   Prove the following *a priori* estimate
   $$\|u_h\|_{L^\infty(\Omega_h)} \le C\|f_h\|_{L^2_h(\Omega_h)}.$$
2. Let V_h be a finite element space consisting of piecewise linears over a mesh of size h. Define the Lagrange interpolant $I_h : C([0,1]) \to V_h$ by
   $$I_h w(x_k) = w(x_k),$$
   where $\{x_k\}$ are the nodes of the mesh. Show that, if $K = [x_k, x_{k+1}]$ is any subinterval, there is a constant, independent of K and h such that, if $v \in C([0,1])$ and it satisfies $v'' \in L^2(K)$, then
   $$\|(I_h v - v)'\|_{L^2(K)} \le Ch\|v''\|_{L^2(K)}.$$
3. Prove the following *discrete maximum principle*: Let $\Omega \subset \mathbb{R}$ be a bounded interval and $I = (0,T]$. Consider a spatial mesh of size h and a temporal one of size τ. Denote by δ_t the forward differencing operator in the t variable and by δ_x and $\bar{\delta}_x$ the forward and backward, respectively, difference operators in the x variable. Show that if $\lambda = \tau/h^2 \le \frac{1}{2}$ and
   $$\delta_t U_j^n - \delta_x\bar{\delta}_x U_j^n \le 0,$$
   for all $(x_j, t_n) \in \Omega \times I$, then the mesh function $U = \{U_j^n\}$ attains its maximum at the discrete parabolic boundary, i.e. at the collection of points
   $$\Gamma_p^n = \{(x_j, 0) \in \bar{\Omega} \times \{0\}\} \cup \{(0, t_k) : k \le n\} \cup \{\{(1, t_k) : k \le n\}$$

CAM PRELIM, JANUARY 2017

1. Numerical Linear Algebra

1.1. A matrix $\mathsf{A} \in \mathbb{C}^{n\times n}$ is called strictly diagonally dominant (SDD) iff

$$|a_{i,i}| > \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{i,k}| , \quad \forall\, i = 1, \ldots, n.$$

a. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be SDD, represented as

$$\mathsf{A} = \begin{bmatrix} \alpha & \mathbf{q}^T \\ \mathbf{p} & \hat{\mathsf{A}} \end{bmatrix},$$

where α is a scalar, $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{n-1}$, and $\hat{\mathsf{A}} \in \mathbb{C}^{(n-1)\times(n-1)}$. After 1 step of Gaussian elimination (without pivoting), A will be reduced to the matrix

$$\begin{bmatrix} \alpha & \mathbf{q}^T \\ \mathbf{0} & \mathsf{B} \end{bmatrix},$$

where $\mathsf{B} \in \mathbb{C}^{(n-1)\times(n-1)}$. Prove that B is SDD.

b. Prove that, if A is SDD, then A is invertible.

1.2. Let $\mathsf{A} = \mathsf{B} + \mathsf{P} \in \mathbb{C}^{n\times n}$ be invertible. To approximate the solution of $\mathsf{A}\mathbf{x} = \mathbf{f}$, starting from an arbitrary $\mathbf{x}_0 \in \mathbb{C}^n$, we apply the following iterative scheme (where we are implicitly assuming that B is invertible)

$$\mathsf{B}\mathbf{x}_{k+1} + \mathsf{P}\mathbf{x}_k = \mathbf{f}.$$

Show that this iterative scheme is convergent if and only if all the roots of

$$\det(\kappa\mathsf{B} + \mathsf{P}) = 0$$

are of modulus strictly less than one. Use this to show that if A is HPD, then Gauss-Seidel method converges.

Hint: You may use without proof that if M is skew Hermitian, then for every $\mathbf{w} \in \mathbb{C}^n$ we have

$$\Re\left(\mathbf{w}^H\mathsf{M}\mathbf{w}\right) = 0,$$

where $\Re z$ denotes the real part of z.

1.3. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD). Suppose $\mathsf{P} \in \mathbb{R}^{n\times m}$, $m \leq n$, is full rank.

a. Show that $\mathsf{A}_C := \mathsf{P}^T\mathsf{A}\mathsf{P}$ is invertible.

b. Define $\mathsf{Q}_\mathsf{A} := \mathsf{P}\mathsf{A}_C^{-1}\mathsf{P}^T\mathsf{A}$. Show that $\mathsf{Q}_\mathsf{A}\mathbf{u} \in R(\mathsf{P})$ is the best approximation of $\mathbf{u} \in \mathbb{R}^n$ with respect to the A-norm, $\|\cdot\|_\mathsf{A}$, which is defined as follows:

$$(\mathbf{v}, \mathbf{w})_\mathsf{A} := \mathbf{v}^T\mathsf{A}\mathbf{w}, \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n; \quad \|\mathbf{w}\|_\mathsf{A} := \sqrt{(\mathbf{w}, \mathbf{w})_\mathsf{A}}, \quad \forall\, \mathbf{w} \in \mathbb{R}^n.$$

In other words, prove that

$$\mathsf{Q}_\mathsf{A}\mathbf{u} = \operatorname{argmin}\left\{\|\mathbf{z} - \mathbf{u}\|_\mathsf{A}^2 \;\middle|\; \mathbf{z} \in R(\mathsf{P})\right\}.$$

1.4. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be symmetric. Suppose the spectrum, denoted $\sigma(\mathsf{A}) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$, has the following ordering

$$|\lambda_1| \geq \cdots \geq |\lambda_{r-1}| > |\lambda_r| > |\lambda_{r+1}| \geq \cdots \geq |\lambda_n| \geq 0.$$

Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be an orthonormal basis of eigenvectors of A, with $\mathsf{A}\mathbf{w}_k = \lambda_k \mathbf{w}_k$, for $k = 1, \ldots, n$. Suppose that $\theta \in \mathbb{R} \setminus \sigma(\mathsf{A})$ is given and $|\lambda_r - \theta| < |\lambda_k - \theta|$, for all $k \neq r$. The inverse iteration is as follows: given $\mathbf{v}_0$, with $\|\mathbf{v}_0\|_2 = 1$ and $\alpha_r := \mathbf{w}_r^T \mathbf{v}_0 \neq 0$, define, for all $m \geq 0$,

$$\mathbf{v}_{m+1} := \frac{(\mathsf{A} - \theta \mathsf{I})^{-1} \mathbf{v}_m}{\left\|(\mathsf{A} - \theta \mathsf{I})^{-1} \mathbf{v}_m\right\|_2}.$$

Prove that $s_m \mathbf{v}_m \to \mathbf{w}_r$, as $m \to \infty$, where $\{s_m\}_{m=1}^{\infty} \subseteq \{-1, 1\}$.

2. Numerical Solution of Nonlinear Equations

2.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and, for some $\xi \in \mathbb{R}$, $f(\xi) = 0$. Assume that, for some $\delta > 0$, $f \in C^2(I_\delta)$, where $I_\delta = [\xi - \delta, \xi + \delta]$, and $f'(\xi) \neq 0$ and $f''(\xi) \neq 0$. If $|\xi - x_0|$ is sufficiently small, prove that the sequence $\{x_k\}$ defined by Steffensen's method,

$$x_{k+1} = x_k - \frac{f(x_k)}{s_k}, \quad s_k = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)},$$

converges quadratically to the root ξ as $k \to \infty$.

3. Numerical Solution of ODEs

3.1. Suppose that $T > 0$, $M \in \mathbb{N}$, $s = T/M$, and $u \in C^{p+1}([0, T])$ satisfies the IVP $u' = f(t, u)$, for $t \in [0, T]$, with $u(0) = u_0$. Suppose that u is approximated by the linear q-step method

$$\sum_{j=0}^{q} a_j w^{k+j} = s \sum_{j=0}^{q} b_j f(t_{k+j}, w^{k+j}),$$

and define

$$C_\ell = \begin{cases} \sum_{j=0}^{q} a_j = 0 & \text{if} \quad \ell = 0 \\ \sum_{j=0}^{q} \left(\frac{j^\ell}{\ell!} a_j - \frac{j^{\ell-1}}{(\ell-1)!} b_j \right) & \text{if} \quad \ell \in \{1, 2, 3, \cdots\} \end{cases}.$$

Show that the scheme is exactly order p iff $C_0 = 0 = C_1 = \cdots = C_p$, but $C_{p+1} \neq 0$.

3.2. a. Prove that no explicit Runga-Kutta method can be A-stable.

b. Consider the implicit Runge-Kutta method given by the tableaux

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Prove that this method is A-stable.

4. Numerical Solution of PDEs

4.1. Suppose that $u \in C^4([0,1])$ is the solution to the one dimensional Poisson problem

$$-\frac{d^2u}{dx^2} = f \quad \text{in} \quad \Omega = (0,1), \qquad u(0) = u(1) = 0.$$

Suppose $m \in \mathbb{N}$, $h = \frac{1}{m+1}$, $x_i = i \cdot h$ and w is the finite difference approximation satisfying

$$-\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} = f(x_i) =: f_i, \quad i = 1, \cdots, m,$$

with $w_0 = w_{m+1} = 0$. Define $e_i := u(x_i) - w_i$, $i = 1, \cdots, m$, and prove that

$$\|e\|_{L_h^2} = \sqrt{h \sum_{i=1}^{m} |e_i|^2} \leq Ch^2,$$

for some constant $C > 0$ that is independent of h. *You may assume the correct form (order) of the local truncation error estimate without proof.*

4.2. Consider the diffusion problem

$$\begin{aligned}
\partial_t u &= \partial_{xx} u, & 0 < x < 1,\ 0 < t \leq T \\
u(0,t) &= \phi_0(t), & 0 \leq t \leq T, \\
u(1,t) &= \phi_1(t), & 0 \leq t \leq T, \\
u(x,0) &= g(x), & 0 \leq x \leq 1,
\end{aligned}$$

where $g(0) = \phi_0(0)$ and $g(1) = \phi_1(0)$ for consistency. Define $h = \frac{1}{m+1}$, $s = \frac{T}{N}$, $\mu = \frac{s}{h^2}$, $x_\ell = \ell \cdot h$, $t_n = n \cdot s$, $g_\ell = g(x_\ell)$, $\phi_i^n = \phi_i(t_n)$, $i = 0, 1$. The forward Euler scheme for this problem is defined as follows:

$$w_\ell^{n+1} = w_\ell^n + \mu\left(w_{\ell-1}^n - 2w_\ell^n + w_{\ell+1}^n\right), \quad 1 \leq \ell \leq m, \quad 0 \leq n \leq N-1,$$

with $w_0^n = \phi_0^n$, $w_{m+1}^n = \phi_1^n$, $0 \leq n \leq N$, and $w_\ell^0 = g_\ell$, $1 \leq \ell \leq m$.

a. Suppose $u \in C^4([0,1] \times [0,T])$ and define $u_\ell^n = u(x_\ell, t_n)$. Prove that

$$u_\ell^{n+1} = u_\ell^n + \mu\left(u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n\right) + s\tau_\ell^n, \quad 1 \leq \ell \leq m, \quad 0 \leq n \leq N-1,$$

where

$$|\tau_\ell^n| \leq C_{4,x} h^2 + C_{2,t} s$$

and $C_{4,x}, C_{t,2} > 0$ are independent of h and s.

b. Define $e_\ell^n := u_\ell^n - w_\ell^n$, $0 \leq \ell \leq m+1$, $0 \leq n \leq N$, $\mathbf{e}^n = [e_1^n, \ldots, e_m^n]^T$, $0 \leq n \leq N$. Suppose that $\mu = \mu_0 \leq \frac{1}{2}$. Prove that

$$\max_{0 \leq n \leq N} \|\mathbf{e}^n\|_\infty \leq Ch^2,$$

where $C > 0$ is independent of h and s.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION. AUGUST 2016

1. Numerical Linear Algebra

1. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be a normal matrix, with spectrum $\sigma(\mathsf{A}) = \{\lambda_1, \dots, \lambda_n\}$. Let $\mathsf{A} = \mathsf{QR}$ be a QR factorization of A. Prove that
$$\min_{1\le j\le n} |\lambda_j| \le |r_{i,i}| \le \max_{1\le j\le n} |\lambda_j| ,$$
for all $i = 1, \dots, n$, where $r_{i,j} = [\mathsf{R}]_{i,j}$.
2. Suppose that $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{n\times n}$ is strictly *column-wise* diagonally dominant, *i.e.*,
$$|a_{k,k}| > \sum_{j\neq k} |a_{j,k}|.$$
   (a) Prove that Gaussian elimination can be preformed to row reduce A without the need for pivoting.
   (b) Can you conclude from part (a) that the matrix is invertible? Why or why not?
3. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be hermitian positive definite (HPD). To solve the system of equations $\mathsf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{b} \in \mathbb{C}^n$ consider the iterative scheme
$$\mathsf{B}\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\tau} + \mathsf{A}\mathbf{x}_k = \mathbf{b}, \qquad k = 0, 1, \cdots ,$$
with $\tau > 0$ and B also HPD. Show that, if $\mathsf{B} - \frac{\tau}{2}\mathsf{A}$ is HPD, then we have $\mathbf{e}_k^H \mathsf{A}\mathbf{e}_k \to 0$ as $k \to \infty$, where $\mathbf{e}_k := \mathbf{x} - \mathbf{x}_k$. Why does this imply convergence?
Hint: $a = \frac{1}{2}(a+b) - \frac{1}{2}(b-a)$.
4. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD) and $\mathbf{0} \neq \mathbf{f} \in \mathbb{R}^n$ be given. To solve the system $\mathsf{A}\mathbf{x} = \mathbf{f}$, we employ the conjugate gradient (CG) method, starting from the initial guess $\mathbf{x}_0 = \mathbf{0}$.
   (a) Suppose that A has only two eigenvalues, $0 < \lambda_1 < \lambda_2$. Prove that, for every $\lambda \in (\lambda_1, \lambda_2)$
$$\|\mathbf{e}_1\|_{\mathsf{A}} \le q(\lambda)\|\mathbf{e}_0\|_{\mathsf{A}}, \quad q(\lambda) := \max_{i=1}^{2} \left|1 - \frac{\lambda_i}{\lambda}\right|.$$
   (b) Prove that
$$\frac{1}{2}(\lambda_1 + \lambda_2) = \mathrm{argmin}_{\lambda\in(\lambda_1,\lambda_2)}\, q(\lambda).$$
   Hint: $\max\{|a|, |b|\} = \frac{1}{2}|a+b| + \frac{1}{2}|a-b|$.
   (c) Use the last fact to show that
$$\|\mathbf{e}_1\|_{\mathsf{A}} \le \frac{\kappa_2 - 1}{\kappa_2 + 1}\|\mathbf{e}_0\|_{\mathsf{A}},$$
   where κ_2 is the spectral condition number of A.

2. Numerical Solution of Nonlinear Equations

1. Let $f \in C^2(\mathbb{R})$ and assume that for all $x \in \mathbb{R}$

$$f'(x) > 0, \qquad f''(x) > 0.$$

   (a) Give an example of a function that satisfies these conditions but has no roots.
   (b) If the function f has roots, how many can it have?
   (c) Assume that f has a root $x_\star$. To approximate it we employ Newton's method, starting from an initial guess x_0. Show that if $x_0 > x_\star$ then $x_0 > x_1 > \cdots > x_k > x_{k+1} > x_\star$, for all $k \geq 1$.
   (d) Use the last step to show that the method converges.

3. Numerical Solution of ODEs

1. Show that, if $z \in \mathbb{C} \setminus \{0\}$ is on the boundary of the linear stability domain of the BDF2 method

$$x^{k+2} - \frac{4}{3}x^{k+1} + \frac{1}{3}x^k = \frac{2}{3}hf(t_{k+2}, x^{k+2}),$$

then $\Re(z) > 0$. Deduce from this that the method is A-stable.

2. Find the range of $a \in \mathbb{R}$ for which the method

$$\eta^{k+2} + (a-1)\eta^{k+1} - a\eta^k = \frac{h}{4}\left((a+3)f(t_{k+2}, \eta^{k+2}) + (3a+1)f(t_k, \eta^k)\right),$$

is consistent and root stable. Apply the method with $a = -1$ to the scalar IVP $y' = y$, $y(0) = 1$ and solve exactly the resulting difference equation, considering the starting values to be $\eta^0 = \eta^1 = 1$. Show theoretically that the numerical solution does not converge as $h \to 0$ and $n \to \infty$.

4. Numerical Solution of PDEs

1. Suppose that u satisfies

$$-u'' + q(x)u = f(x), \ x \in (0,1), \qquad u \text{ is 1-periodic},$$

where q and f are smooth periodic functions and $q(x) \geq q_0 > 0$, for all $x \in \mathbb{R}$.
   (a) Over a uniform mesh of size h construct a finite difference scheme that is second order consistent. (You don't need to prove the order of the local truncation error.)
   (b) Prove the following ℓ^∞ stability result:

$$\|w\|_{\ell^\infty} \leq \frac{1}{q_0}\|f\|_{\ell^\infty}.$$

   where w is the periodic grid function approximating u.
   Hint: Consider the approximation scheme at the maximum and minimum values of the grid function w.

2. Consider the boundary value problem

$$-u'' = f(x), \ x \in (0,1), \qquad u(0) = u(1) = 0.$$

Given $N \in \mathbb{N}$ we construct a mesh with nodes $x_i = i/N$ with $i = 0, \ldots, N$. Further, assume that f, restricted to the interval $(x_i, x_{i+1}]$ is constant. Show that the standard finite element scheme with piecewise linear basis functions coincides with the finite difference scheme

$$-\frac{U^{i+1} - 2U^i + U^{i-1}}{h^2} = \frac{1}{2}(F^{i-1/2} + F^{i+1/2}),$$

where $h = 1/N$ and $F^{i+1/2}$ is the value of f on the interval $(x_i, x_{i+1}]$.

3. Suppose that $u : [0,1] \times [0,T] \to \mathbb{R}$ is a smooth solution to the linear transport equation $u_t + au_x = 0$ in $(0,1) \times (0,T)$, with periodic boundary conditions, $u(0,t) = u(1,t)$, for all $t \in [0,T]$, and smooth, periodic initial data $u(\cdot, 0) = g$. Suppose that u is approximated by the upwind scheme, with time step size $s = T/N$, and space step size $h = 1/M$, where M and N are positive integers.
   (a) Prove that, under the appropriate CFL condition – which you must state – the scheme is stable in the norm $\|\cdot\|_{\ell^\infty}$.
   (b) Prove that, under the same CFL condition, the scheme is convergent in the norm $\|\cdot\|_{\ell^\infty}$, *i.e.*, for any integer n, $0 \le n \le N$, there is a constant $C > 0$, which is independent of s and h, such that

$$\|e^n\|_{\ell^\infty} \le C \cdot T(s+h),$$

   where e^n is the error grid function. (You may assume the appropriate form of the local truncation error without proof.)

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION: JANUARY 2016

1. Numerical Linear Algebra

1. For $A = [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ define

$$\mathcal{M}(A) = n \cdot \max_{1\leq i,j\leq n} |a_{ij}|.$$

   (i) Though it is not a *matrix norm*, show that $\mathcal{M}(\cdot)$ is a *vector norm* over the linear space $\mathbb{R}^{n\times n}$.
   (ii) Show that for $p \in \{1, 2, \infty\}$ the following inequalities hold:

$$\frac{1}{n}\mathcal{M}(A) \leq \|A\|_p \leq \mathcal{M}(A).$$

2. Suppose that $A \in \mathbb{R}^{n\times n}$ has the property that

$$a_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \qquad a_{ij} < 0,\ i \neq j.$$

   Show that, in this case A^{-1} exists and contains only non-negative elements. *Hint*: Recall the procedure to invert a matrix using Gaussian elimination.

3. Suppose $A \in \mathbb{R}^{n\times n}$. To obtain the solution of $Ax = f$ we apply the Gauss-Seidel method and obtain a sequence $\{x_k\}_{k\geq 0}$ of approximate solutions. Assume that the matrix A is such that there is $q \in (0, 1)$ for which

$$q|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|.$$

   Show that the following error estimate holds:

$$\|x - x_k\|_\infty \leq q^k \|x - x_0\|_\infty.$$

4. Suppose $A \in \mathbb{R}^{n\times n}$ and $A = A^{\mathsf{T}}$. For simplicity, assume that the eigenvalues of A are ordered as follows:

$$0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|.$$

   Assume that λ_n and its corresponding eigenvalue φ_n are known. Show that if

$$(x^0, \varphi_n) \neq 0, \qquad x^{k+1} = Ax^k, \qquad y^{k+1} = x^{k+1} - \lambda_n x^k,$$

   then

$$\frac{(y^{k+1}, y^k)}{(y^k, y^k)} = \lambda_{n-1} + \mathcal{O}\left(\left|\frac{\lambda_{n-2}}{\lambda_{n-1}}\right|^{2k}\right).$$

2. Numerical Solution of Nonlinear Equations

1. By $B(x, y, r) \subset \mathbb{R}^2$ denote the open ball of radius $r > 0$ centered at (x, y). Suppose that for some $r > 0$, $f, g : B(x_\star, y_\star, r) \to \mathbb{R}$ are nonlinear, twice continuously differentiable functions with

$$f(x_\star, y_\star) = 0, \qquad g(x_\star, y_\star) = 0.$$

Consider the Gauss-Seidel-like iterative scheme: given $(x^k, y^k) \in B(x_\star, y_\star, r)$, find $(x^{k+1}, y^{k+1}) \in \mathbb{R}^2$ such that

$$f(x^{k+1}, y^k) = 0, \qquad g(x^{k+1}, y^{k+1}) = 0.$$

(i) Establish an iteration error equation of the form

$$\begin{bmatrix} f_x(x_\star, y_\star)e_x^{k+1} + f_y(x_\star, y_\star)e_y^k \\ g_x(x_\star, y_\star)e_x^{k+1} + g_y(x_\star, y_\star)e_y^{k+1} \end{bmatrix} = \vec{R}^{k+1},$$

where $e_x^k := x_\star - x^k$ and $e_y^k = y_\star - y^k$, giving a precise expression for the 'remainder' term, $\vec{R}^{k+1}$.

(ii) Using problem 1.3 as a guide, give sufficient conditions for the convergence of the scheme.

3. Numerical Solution of ODEs

Consider the solution of the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0.$$

The following two questions are about the linear multistep method

$$\sum_{j=0}^{k} a_j x_{n-k+j} = h \sum_{j=0}^{k} b_j f_{n-k+j},$$

where $f_\ell = f(\ell \cdot h, x_\ell)$ and $h > 0$ is the step size.

1. Define

$$d_0 = \sum_{i=0}^{k} a_i \qquad d_j = \sum_{i=0}^{k} \left(\frac{i^j}{j!} a_i - \frac{i^{j-1}}{(j-1)!} b_i \right) \quad j \geq 1.$$

Show that $d_0 = \cdots = d_m = 0$ if and only if the local order of truncation of the method is m.

2. Recall that, to the linear multistep method, we can also associate the polynomials

$$p(z) = \sum_{j=0}^{k} a_j z^j, \qquad q(z) = \sum_{j=0}^{k} b_j z^j.$$

Show that Milne's method, for which

$$p(z) = z^2 - 1, \qquad q(z) = \frac{1}{3}z^2 + \frac{4}{3}z + \frac{1}{3},$$

is (root) stable, consistent, and convergent.

4. Numerical Solution of PDEs

1. Consider the two-point boundary value problem

$$-u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

Suppose that the solution u is approximated via the finite element method using a uniform mesh, of mesh size h, and a finite element space composed of piecewise linear functions.

   (i) Denoting the finite element approximation by u_h, prove the fundamental Galerkin orthogonality: $\int_0^1 (u_h - u)' v_h' \, dx = 0$, for all v_h in the piece-wise linear finite element space.

   (ii) Using (i), show that the method is *exact* at the nodes x_k of the mesh, that is, $u_h(x_k) = u(x_k)$.

2. Consider the Lax Friedrichs scheme,

$$w_\ell^{n+1} = \frac{1}{2}\left(w_{\ell-1}^n + w_{\ell+1}^n\right) - \frac{\mu}{2}\left(w_{\ell+1}^n - w_{\ell-1}^n\right), \quad \mu = \frac{a\,s}{h},$$

for approximating solutions to the Cauchy problem for the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, where $a > 0$. Here $h > 0$ is the space step size, and $s > 0$ is the time step size.

   (a) Prove that, if $s = C_1 h$, where C_1 is a fixed positive constant, then the local truncation error, τ_ℓ^n, satisfies the estimate

$$|\tau_\ell^n| \le C_0\,(s + h)\,,$$

   where $C_0 > 0$ is a constant independent of s and h.

   (b) Use the von Neumann analysis to show that the Lax-Friedrichs scheme is stable provided the CFL condition

$$0 < \mu = \frac{a\,s}{h} \le 1$$

   holds.

Computational Mathematics Preliminary Exam
August 12, 2015

There are 9 problems, some with parts. Complete as many problems or parts of problems as possible. Good luck!

1. Let B be a singular symmetric matrix with $\|B\|_2 = 1$. Given $\epsilon > 0$, construct an invertible matrix A such that $\|A - B\|_2 < \epsilon$ and determine an upper or lower bound on $\kappa_2(A)$ that depends on ϵ.

2. Let A be an $n \times n$ symmetric and positive definite matrix with Cholesky factorization $A = LL^T$. Let I_n be the $n \times n$ identity, $\alpha > 0$ and construct the $2n \times 2n$ block matrix
$$B = \begin{pmatrix} A & \alpha I_n \\ \alpha I_n & A \end{pmatrix}.$$
Under what condition(s) on α is B positive definite?
Under those conditions, determine B's Cholesky Factorization.

3. Let A_1 and A_2 be $n \times n$ real matrices with A_1 invertible, and b_1 and b_2 be n-vectors. Consider the following iterative method: Given (x^0, y^0), for $k = 0, 1, 2, \ldots$
$$\begin{aligned} A_1 x^{k+1} &= b_1 + A_2 y^k \\ A_1 y^{k+1} &= b_2 - A_2 x^k. \end{aligned}$$
If this method converges, what does (x^k, y^k) converge to?
Under what condition(s) on A_1, A_2, b_1 and b_2, does this method converge for any choice of (x^0, y^0)?

4. Let $f : \mathrm{R} \to \mathrm{R}$ be twice continuously differentiable. Suppose $f(x^*) = 0$ and $f'(x^*) > 0$. Show that there's an $\epsilon > 0$ such that if $|x_0 - x^*| < \epsilon$ then the following iteration converges to x^*:
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_m)},$$
where $m \le k$ is chosen such that $|f'(x_m)| = \max_{j \le k} |f'(x_j)|$.

5. Let A be an $n \times n$ symmetric matrix and λ its largest eigenvalue with corresponding eigenvector v. Assume no other eigenvalue of A has the same magnitude as λ.
Let $x^0 = v + \delta u$ for some small scalar δ and a non-zero vector u with $v^T u = 0$, and consider the following iterative scheme for $k = 0, 1, 2, \ldots$:
$$\begin{aligned} y^{k+1} &= A x^k \\ \mu_{k+1} &= \frac{y^{k+1} \cdot x^0}{x^k \cdot x^0} \\ x^{k+1} &= \frac{y^{k+1}}{\|y^{k+1}\|_2}. \end{aligned}$$
Under what condition(s) does μ_k converges to λ?

6. For $0 < \delta < 1$, an ODE solver is δ-damping if when the method is applied to solve $y' = \lambda y$, $|y_{n+1}| \leq \delta |y_n|$ for all n as $\lambda h \to -\infty$.

   Show that implicit trapezoid method

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$

   is A-stable but that it is not δ-damping.

7. Consider the general s-step Backward Differentiation Formula (BDF) for solving the ODE $y' = f(t, y)$:

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}).$$

   Determine coefficients for a 2-step method that is at least 2nd order.

   For this method, determine whether is A-stable or not.

8. Consider the solution of

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

   via the scheme

$$\frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1}) = f(x_j), \qquad j = 1, \dots, m,$$

   where $h = \frac{1}{m+1}$, $x_j = jh$, and U_j is our approximation to the solution $u(x_j)$ (so $U_0 = U_{m+1} = 0$). Show that this scheme is convergent in the 2-norm.

9. Consider the Lax-Wendroff scheme

$$U_\ell^{n+1} = U_\ell^n + \frac{a^2k^2}{2h^2}\left(U_{\ell-1}^n - 2U_\ell^n + U_{\ell+1}^n\right) - \frac{ak}{2h}\left(U_{\ell+1}^n - U_{\ell-1}^n\right),$$

   for approximating the solution of the Cauchy problem for the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, $0 < t < T$ where $a > 0$. We have $k = \frac{T}{N}$, h is the spacing for the grid $\{x_\ell\}$, and U_ℓ^n is the approximation of $u(nk, x_\ell)$.

   Use Von Neumann's method to show that the Lax-Wendroff scheme is stable provided the CFL condition

$$\frac{ak}{h} \leq 1$$

   is enforced.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION.
MONDAY AUGUST 11, 2014

1. Numerical Linear Algebra

(1) Let $A_i \in \mathbb{R}^{n \times n}$ with $i = 1, 2$ be two SPD matrices that commute, that is $A_1 A_2 = A_2 A_1$. Define $A = A_1 + A_2$. To solve the linear system of equations $Ax = f$ consider the following iterative scheme:

$$(I + \tau A_1)\frac{x^{k+1/2} - x^k}{\tau} + Ax^k = f,$$
$$(I + \tau A_2)\frac{x^{k+1} - x^{k+1/2}}{\tau} + Ax^{k+1/2} = f,$$

where $\tau > 0$ is a user defined parameter.

(a) Write this scheme in the form

$$B\frac{x^{k+1} - x^k}{\alpha} + Ax^k = f$$

and clearly identify the value of α and the iterator B. *Hint:* Try adding and subtracting the steps.

(b) Is B invertible? Justify your answer.

(c) From this expression find the equation that controls the error $e^k = x - x^k$.

(d) Show that with the given assumptions $[x, y] = (A_1 A_2 x, y)$ is an inner product.

(e) Show that for every $\tau > 0$ the sequence $y^{k+1/2} = \frac{1}{2}(x^{k+1} + x^k)$ converges to x. *Hint:* Take the inner product of the equation that controls the error with $e^{k+1} + e^k$ and add over k.

(2) Show that for every $x \in \mathbb{R}^n$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

Use these identities to show that for any $A \in \mathbb{R}^{n \times n}$

$$\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_\infty \leq \sqrt{n}\|A\|_2$$

and that, if A is nonsingular,

$$\frac{1}{n} \leq \frac{\kappa_\infty(A)}{\kappa_2(A)} \leq n.$$

(3) A projector is a square matrix P that is *idempotent*, that is $P^2 = P$. An orthogonal projector is one that satisfies $P = P^\star$. Show that:

(a) If P is an orthogonal projector then $I - 2P$ is unitary.

(b) If P is a projector and $P \neq 0$ then $\|P\|_2 \geq 1$ with equality if P is an orthogonal projector.

2. Numerical Solution of Nonlinear Equations

(4) Consider the following relaxation method for the solution of $f(x)=0$:

$$\frac{x^{k+1}-x^k}{\alpha}+f(x^k)=0.$$

Assume that $0<m\le f'(x)\le M$. Provide sufficient conditions on α for convergence.

3. Numerical Solution of ODEs

(5) Show that there is no two stage third order explicit Runge-Kutta scheme.

(6) To approximate the solution to $y'(t) = f(t,y(t))$ we use the following multistep method

$$\frac{1}{12h}\left(25y^{n+1}-48y^n+36y^{n-1}-16y^{n-2}+3y^{n-3}\right)=f^{n+1}.$$

Determine its order. Is it A-stable?

4. Numerical Solution of PDEs

(7) Consider the boundary value problem

$$-(a(x)u')'+c(x)u=f \text{ in } (0,1) \quad u(0)=u(1)=0.$$

Develop a finite difference approximation of order $\mathcal{O}(h^2)$. You must show that this is indeed the order of the scheme. Notice that the coefficients are variable.

(8) Consider the *Dufort-Frankel* scheme

$$\frac{1}{2\tau}(U_j^{n+1}-U_j^{n-1})=\frac{1}{h^2}(U_{j+1}^n-2\bar{U}_j^n+U_{j-1}^n), \qquad \bar{U}_j^n=\frac{1}{2}(U_j^{n+1}+U_j^{n-1})$$

to approximate the heat equation. Show that if $\mu=\frac{\tau}{h}$ tends to zero, then this scheme is consistent. If, instead, μ is constant, then the scheme is consistent with the equation

$$\mu^2u_{tt}+u_t-u_{xx}=0.$$

(9) A general linear explicit finite difference scheme can be written in the form

$$U_j^{n+1}=\sum_{p\in\mathcal{P}}a_pU_{j+p}^n,$$

where $\mathcal{P}$ is a finite subset of $\mathbb{Z}$. We say that the scheme *reproduces the constant state* if, whenever $U^n\equiv 1$, we obtain that $U^{n+1}\equiv 1$.

(a) Show that if the scheme reproduces the constant state, then $\sum_{p\in\mathcal{P}}a_p=1$.

(b) A scheme is max-norm preserving if $\|U^{n+1}\|_{L_h^\infty}\le\|U^n\|_{L_h^\infty}$. Show that if a scheme reproduces the constant state and is max-norm preserving, then $a_p\ge 0$ for all $p\in\mathcal{P}$.

(c) Assume that this scheme is used to approximate the transport equation $u_t+u_x=0$ with initial data $u(x,0)=e^{\imath x}$. Find an expression for the error in the L_h^2-norm after one step that depends only on the coefficients $\{a_p\}_{p\in\mathcal{P}}$, the mesh size h and the ratio $\mu=\tau/h$.

Notation: For a mesh function U the L_h^2 and L_h^∞ norms are given by

$$\|U\|_{L_h^2}^2=h\sum_j|U_j|^2, \qquad \|U\|_{L_h^\infty}=\max_j\{|U_j|\}.$$

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION
FRIDAY JANUARY 3, 2014

NAME: ____

I. Numerical Linear Algebra

1. Let $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = r$. If A has the SVD $A = U\Sigma V^*$, the *Moore-Penrose pseudoinverse* of A is defined by

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger = \operatorname{diag}\{\sigma_1^{-1}, \dots, \sigma_r^{-1}, 0, \dots, 0\} \in \mathbb{R}^{n\times m}$. Show the following:

   (a) If A^{-1} exists, then $A^\dagger = A^{-1}$.

   (b) If A has full rank, then $A^\dagger = (A^*A)^{-1}A^*$.

   (c) $AA^\dagger A = A$.

   (d) $A^\dagger AA^\dagger = A^\dagger$.

2. Let $A \in \mathbb{C}^{m\times m}$ and let a_j be its j-th column. Prove *Hadamard's inequality*

$$|\det(A)| \le \prod_{j=1}^{m} \|a_j\|_2.$$

3. Suppose $A, B \in \mathbb{C}^{m\times m}$ are Hermitian positive definite. Assume there are constants $\gamma_1, \gamma_2 > 0$ such that, for all $x \in \mathbb{C}^m$,

$$\gamma_1 x^* Bx \le x^* Ax \le \gamma_2 x^* Bx.$$

Consider the so-called *energy norm* with respsect to B:

$$\|x\|_B^2 = x^* Bx, \quad \forall x \in \mathbb{C}^m.$$

Show that

$$\kappa_B(B^{-1}A) \le \frac{\gamma_2}{\gamma_1},$$

where by κ_B we denote the condition number with respect to the subordinate matrix norm generated by the energy norm.

4. Let $A \in \mathbb{R}^{n\times n}$ matrix and $b \in \mathbb{R}^n$. Given a linear system $Ax = b$, consider the following iterative method:

$$x_{k+1} = x_k + \alpha r_k,$$

where $r_k := b - Ax_k$ is the residual, $x_0 \ne A^{-1}b$ is arbitrary, and α is a scalar parameter to be determined.

(a) Show that if all the eigenvalues of A have positive real part, then there will be some real α such that the method converge for any starting vector x_0.

(b) Show that if some eigenvalues of A have negative real part and some have positive real part, then there is no real α for which the iterations converge.

II. Numerical Solutions of Nonlinear Equations

5. Consider the nonlinear equation $e^x = \sin x$.

   (a) Show that there is a solution $x_* \in (-\frac{5}{4}\pi, -\pi)$.

   (b) Consider the following iterative methods: (1): $x_{k+1} = \ln(\sin x_k)$ and (2) $x_{k+1} = arcsin(e^{x_k})$. What can you say about the local convergence of each of these methods for x_* as in (a), and their convergence order?

   (c) For x_* as in (a), please give a method which is quadratically convergent. Justify why this method is quadratically convergent (you can use a theorem here, but you need to give precisely the condition in that theorem).

III. Numerical Solutions of ODEs

6. (a) Outline the derivation of (explicit) Adams-Bashforth methods for the numerical solution of the initial value problem

$$y' = f(t, y), \quad y(x_0) = y_0,$$

   and derive the Adams-Bashforth formula

$$y_{n+1} = y_n + h\left[-\frac{1}{2}f_{n-1} + \frac{3}{2}f_n\right], \qquad f_n = f(t_n, y_n).$$

   (b) Analyze the above method, to be more specific, find the local truncation error and prove convergence.

7. This problem is about choosing between a specific single-step and a specific multi-step methods for solving the ODE:

$$y' = f(t, y).$$

   (a) Write the trapezoidal method and find its local truncation error.

   (b) Show that the local truncation error for the following multistep method is of the same order as in (a):

$$y_{n+1} = 2y_n - y_{n-1} - hf_{n-1} + hf_n, \qquad f_n = f(t_n, y_n).$$

   (c) What could be said about the global convergence rate for these two methods? Justify your conclusions for both methods.

IV. Numerical Solutions of PDEs

8. Consider the problem

$$
\begin{aligned}
&u_t - u_{xx} - e^u = 0, \qquad && \text{in } [0,1] \times [0,T], \\
&u(x,0) = u(x,1) = 0, \qquad && \text{in } [0,T], \\
&u(x,0) = u_0(x), \qquad && \text{in } [0,1].
\end{aligned}
$$

We want to design a numerical method by the method of line discretization, with trapezoidal rule to discretize time and the Galerkin method to discretize space. Write down this method.

9. Approximate the heat equation $u_t = u_{xx}$ by the method

$$
\frac{u_j^{n+1} - u_j^n}{k} = \theta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + (1-\theta)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2},
$$

where $k = \Delta t$, $h = \Delta x$ and $0 \le \theta \le 1$.

(a) Show that for any θ the scheme is consistent and has the local truncation error of $O(k + h^2)$.

(b) Find the value of θ that yields local truncation error of $O(k^2 + h^2)$.

(c) Assuming that $\lambda = k/h^2 = const$, find the value of θ that yields local truncation error of $O(h^4)$.

(d) Apply the Von Neumann analysis on this scheme with the choice of θ in (b), to study the stability of this scheme (i.e. under which condition, this scheme will be stable).

Numerical Mathematics Preliminary Examination
Friday August 16, 2013

NAME: ______________________________

I. Numerical Linear Algebra

1. Let $A \in \mathbb{R}^{m\times n}$ matrix and $b \in \mathbb{R}^n$ with $m > n$.
   (a) Assume that the rank$(A) = n$. Derive the equations that determine the solution to
   $$x = \arg\min_{z\in\mathbb{R}^m} \|Az - b\|_2^2.$$
   (b) Outline a procedure for obtaining the solution to these equations that avoids problems due to ill-conditioning that may occur if one uses Gaussian elimination on the equations directly.

2. Let $A \in \mathbb{R}^{n\times n}$ be a matrix with $\det(A) \neq 0$. Prove that
   $$\frac{1}{\|A^{-1}\|_2} = \min_{\det(B)=0} \|A - B\|_2.$$

3. Let A be a real symmetric positive-definite matrix. Given a linear system $Ax = b$, consider the following iterative method:
   $$x_{k+1} = x_k + \alpha_k r_k,$$
   where $r_k := b - Ax_k$ is the residual, $x_0 \neq A^{-1}b$ is arbitrary, and α_k is a scalar parameter to be determined.
   (a) Derive an expression for α_k such that $\|r_{k+1}\|_2$ is as small as possible.
   (b) Is this expression always well-defined and nonzero?
   (c) Show that with this choice of α_k,
   $$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \left(1 - \frac{\lambda_{\min}(A)}{\lambda_{\max}(A)}\right)^{k/2}.$$

4. Let
   $$A := \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}, \quad S := \begin{bmatrix} A_1 & O \\ O & A_3 \end{bmatrix},$$
   where $A_1, A_2, A_3, O \in \mathbb{R}^{d\times d}$, and O is the zero matrix. Suppose A is SPD.
   (a) Prove that A_1, A_3 and S are also SPD.
   (b) Let $L_1L_1^T = A_1$, $L_3L_3^T = A_3$, and $BB^T = S$ be respective Cholesky factorizations. Writing B in terms of L_1 and L_3, prove that
   $$C := B^{-1}AB^{-T} = \begin{bmatrix} I & F \\ F^T & I \end{bmatrix}, \quad \text{where} \quad F := L_1^{-1}A_2L_3^{-T}.$$

Hint: Start by writing $A = BB^T + \begin{bmatrix} 0 & A_2 \\ A_2^T & 0 \end{bmatrix}$.

(c) Let λ_i, $i = 1, \dots 2d$, and μ_j, $j = 1, \dots, d$, be the eigenvalues of C and FF^T, respectively. Prove, that, with the appropriate numbering,

$$\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{k+d} = 1 + \sqrt{\mu_k}, \quad k = 1, \dots, d.$$

Hint: Calculate $(C - I)^2$ and find the eigenvalues thereof.

(d) Let $\mathrm{rank}(A_2) = r < d$. Prove that $\mathrm{rank}(FF^T) = r$.

(e) Deduce that C has at most $2r + 1$ distinct eigenvalues. In what number of iterations is the CG algorithm (with exact arithmetic) guaranteed to converge if applied to solve $Cx = b$?

II. Numerical Solutions of Nonlinear Equations

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with a simple root at $x = x^*$. Suppose Newton's method is applied where the initial iterate, x^0, is sufficient close to x^*. Let x^k and x^{k+1} be two successive approximate roots. Explain why $|x^{k+1} - x^k|$ is a good approximation to the error $|x^k - x^*|$.

III. Numerical Solutions of ODEs

6. Consider the following general 2-stage explicit Runge-Kutta method for advancing the solution of $dy/dt = F(y)$ with timestep h,

$$\begin{aligned} y^* &= y^n + \alpha h F(y^n), \\ y^{n+1} &= y^n + \beta h F(y^n) + \gamma h F(y^*). \end{aligned}$$

(a) Derive conditions on the coefficients α, β, and γ that insure that the method has at least first order local truncation error.

(b) Assuming that the coefficients of the method are selected so that it is first order, derive the expression that determines the linear stability region for the method.

7. (a) Write down the trapezoidal rule for $y' = f(t, y)$.

(b) Derive an expression for its local truncation error.

(c) Prove the convergence of the trapezoidal rule, assuming the usual conditions on f. (Hint: you can use the following result directly without proving it: if $|e_{n+1}| \le a|e_n| + ch^r$, one has $|e_n| \le ch^{r-1}(a^n - 1)$.)

(d) Find the linear stability domain of the method, and determine whether or not it is A-stable.

IV. Numerical Solutions of PDEs

8. Consider the PDE problem

$$\begin{aligned} u_t = u_{xx} - u, &\quad \text{for} \quad x \in (0,1),\ t \in (0,T), \\ u(0,t) = u(1,t) = 0 &\quad \text{for} \quad t \in (0,T), \\ u(x,0) = g(x) &\quad \text{for} \quad x \in (0,1). \end{aligned}$$

   (a) Using the notation $h = \frac{1}{M+1}$ and $k = \frac{T}{N}$, write a numerical scheme that is second order (in both space and time) and unconditionally stable in the following sense: $\|U^{n+1}\|_{2,h} \le \|U^n\|_{2,h}$, for all $0 \le n \le N-1$ and for any h and k, where $\|U\|_{2,h} := \sqrt{h \sum_{i=1}^{M} U_i^2}$, and U_i^n approximates the true solution $u(i \cdot h, n \cdot k)$, for $0 \le i \le M+1$ and $0 \le n \le M$. Justify your answer on the unconditionally stability. (no need to verify the second order local truncation error).

   (b) Give sufficient conditions on h and k so that $\|U^{n+1}\|_\infty \le \|U^n\|_\infty$ is guaranteed for all $0 \le n \le N-1$.

9. For the Cauchy problem of the advection problem

$$u_t + u_x = 0,$$

we consider the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n + \frac{k^2}{2h^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) - \frac{k}{2h}\left(u_{j+1}^n - u_{j-1}^n\right),$$

for approximating its solution. Here $h > 0$ is the space step size, and $k > 0$ is the time step size. Let $\mu = k/h$.

   (a) Prove that, if μ is constant and $\mu \le 1$, then the local truncation error T_j^n satisfies the estimate

$$|T_j^n| \le C_0\left(k^2 + h^2\right),$$

   where $C_0 > 0$ is a constant independent of k and h. (Note: depending on how the local truncation error is defined, it could become:

$$|T_j^n| \le C_0\left(k^2 + h^2\right) k.$$

   (b) Use the von Neumann analysis to show that the Lax-Wendroff scheme is stable provided the CFL condition

$$0 < \mu = \frac{k}{h} \le 1$$

   holds.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION
FRIDAY, JANUARY 4, 2013

NAME: ____

I. Numerical Linear Algebra

1. Let A be an $m \times n$ matrix with real entries and $\mathbf{b} \in \mathbb{R}^n$ with $m > n$. Assume that $\text{rank}(\mathsf{A}) = n$. Derive the equation that determines the solution to

$$\mathbf{z} = \arg\min \left\{ \|\mathsf{A}\mathbf{x} - \mathbf{b}\|_2^2 \mid \mathbf{x} \in \mathbb{R}^m \right\}.$$

Argue that the solution $\mathbf{z} \in \mathbb{R}^m$ must be unique.

2. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and let $\sigma \in \mathbb{R}$. Define $\mathsf{H}(\mathbf{u}, \mathbf{v}, \sigma) := \mathsf{I} - \sigma \mathbf{u}\mathbf{v}^T$, where I is the $m \times m$ identity matrix.

   (a) Find all nonzero values of σ for which $\mathsf{H}(\mathbf{u}, \mathbf{u}, \sigma)$ is orthogonal. For such σ, determine all the eigenvalues and the corresponding eigenvectors of $\mathsf{H}(\mathbf{u}, \mathbf{u}, \sigma)$.

   (b) Let $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{x} \neq \mathbf{0}$. Describe how to choose a vector $\mathbf{u} \in \mathbb{R}^m$ such that $\mathsf{H} = \mathsf{H}(\mathbf{u}, \mathbf{u}, \sigma)$ has the property that $\mathsf{H}\mathbf{x}$ is a multiple of $\hat{\mathbf{e}}^{(1)} = (1, 0, 0, \cdots, 0)^T$, where σ is as defined in (a).

3. Let $\mathsf{A} \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Let $\mathbf{b} \in \mathbb{R}^n$. Consider solving $\mathsf{A}\mathbf{x} = \mathbf{b}$ using the stationary iterative method given by

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \mathsf{B}^{-1}(\mathbf{b} - \mathsf{A}\mathbf{x}^{(n)}),$$

where $\mathsf{B} \in \mathbb{R}^{n \times n}$ has an easily computable inverse. Suppose that $\mathsf{B} + \mathsf{B}^T - \mathsf{A}$ is positive definite. Let $\mathbf{e}^{(n)} := \mathbf{x}^{(n)} - \mathbf{x}$ be the error of the n-th iteration. Show that each step of this method reduces the A-norm of $\mathbf{e}^{(n)}$, whenever $\mathbf{e}^{(n)} \neq \mathbf{0}$. Recall, the A-norm of any $\mathbf{y} \in \mathbb{R}^n$ is defined via

$$\|\mathbf{y}\|_{\mathsf{A}} \equiv \sqrt{\mathbf{y}^T \mathsf{A}\mathbf{y}}.$$

4. Suppose that A is an upper triangular, nonsingular matrix. Show that both Jacobi and Gauss-Seidel iterations always converge when used to solve $\mathsf{A}\mathbf{x} = \mathbf{b}$, and, moreover, they will converge in finitely many steps.

II. Numerical Solutions of Nonlinear Equations

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined via

$$f(x, y) = \begin{bmatrix} x^2 - 2x + y \\ 2x - y^2 - 1 \end{bmatrix}.$$

Observe that f has the zero

$$\mathbf{x}_* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider the iteration

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathsf{A} f(\mathbf{x}_n), \qquad \mathsf{A} = \begin{bmatrix} 1 & 1/2 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

(a) Prove $\mathbf{x}_n \to \mathbf{x}_*$, provided $\mathbf{x}_0$ is sufficiently close to $\mathbf{x}_*$.

(b) Show that the convergence is at least quadratic.

(c) Is the iteration (1) equivalent to Newton's method?

III. Numerical Solutions of ODEs

6. Applying a p-stage explicit Runge-Kutta method to approximate the solution of the differential equation $y' = \lambda y$, $y(0) = 1$ results in the scheme

$$y_{n+1} = r(\lambda h) y_n, \quad r(z) = \sum_{k=0}^{p} d_k z^k, \quad y_0 = 1.$$

(a) Show that if the method has order p then,

$$r(z) = \sum_{k=0}^{p} \frac{z^k}{k!},$$

i.e., $d_k = \frac{1}{k!}$, $0 \le 1 \le p$.

(b) Show that no explicit p-stage, order-p Runge-Kutta scheme is A-stable.

7. Show that the 2-step (implicit) Adams-Moulton method

$$\eta_{i+2} - \eta_{i+1} = h \left[\frac{5}{12} f(x_{i+2}, \eta_{i+2}) + \frac{8}{12} f(x_{i+1}, \eta_{i+1}) - \frac{1}{12} f(x_i, \eta_i) \right]$$

is third-order. Is the method convergent? Give a detailed analysis to support your last answer.

IV. Numerical Solutions of PDEs

8. Consider the following boundary-value problem: $-\frac{d^2u}{dx^2}(x) = f(x)$, for $x \in (0,1)$, with $u(0) = u_0$, $u(1) = u_1$. A finite difference approximation scheme is given by

$$\frac{-w_{i-1} + 2w_i - w_{i+1}}{h^2} = f_i =: f(x_i), \quad 1 \le i \le m,$$
$$w_0 = u_0, \quad w_{m+1} = u_1,$$

where $h = \frac{1}{m+1}$, $x_i = i \cdot h$, and w_i is the approximation to $u(x_i)$. Suppose that $f(x) \le 0$ for all $x \in (0,1)$. Prove that the approximation w_i satisfies a discrete maximum principle, namely, (i)

$$\max\{u_0, u_1\} \ge w_i,$$

for all $1 \le i \le m$, and (ii) if $\max\{u_0, u_1\} = w_i$, for some $1 \le i \le m$, then $w_i = \alpha$, for all $0 \le i \le m+1$, where α is an appropriate constant.

9. Consider the following linear reaction-diffusion problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad & \text{for} \quad 0 < x < 1, \quad 0 < t \le T, \\
u(0,t) = 0 = u(1,t) \quad & \text{for} \quad 0 \le t \le T, \\
u(x,0) = g(x) \quad & \text{for} \quad 0 \le x \le 1.
\end{aligned}$$

The 'Crank-Nicolson' scheme for this problem is written as

$$w_\ell^{n+1} = w_\ell^n + \frac{\mu}{2}\left(w_{\ell-1}^{n+1} - 2w_\ell^{n+1} + w_{\ell+1}^{n+1} + w_{\ell-1}^n - 2w_\ell^n + w_{\ell+1}^n\right) - \frac{s}{2}\left(w_\ell^{n+1} + w_\ell^n\right),$$

where $h = \frac{1}{m+1}$, $s = \frac{T}{N}$, and $\mu = \frac{s}{h^2}$. Prove that this scheme is convergent in the sense that, for any $0 \le n \le N$,

$$\|\mathbf{e}^n\|_{2,h} := \sqrt{h\sum_{\ell=1}^{m} (e_\ell^n)^2} \le C(h^2 + s^2),$$

where C is independent of space step size h and the time step size s. Clearly state the restrictions that you impose on s and h, if any. You may assume that the local truncation error has the appropriate form you need, without proof.

10. Consider the Lax Friedrichs scheme,

$$w_\ell^{n+1} = \frac{1}{2}\left(w_{\ell-1}^n + w_{\ell+1}^n\right) - \frac{\mu}{2}\left(w_{\ell+1}^n - w_{\ell-1}^n\right), \quad \mu = \frac{a\,s}{h},$$

for approximating solutions to the Cauchy problem for the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, where $a > 0$. Here $h > 0$ is the space step size, and $s > 0$ is the time step size.

(a) Prove that, if $s = C_1 h$, where C_1 is a fixed positive constant, then the local truncation error satisfies the estimate

$$|T_\ell^n| \le C_0\,(s + h),$$

where $C_0 > 0$ is a constant independent of s and h.

(b) Use the von Neumann analysis to show that the Lax-Friedrichs scheme is stable provided the CFL condition

$$0 < \mu = \frac{a\,s}{h} \le 1$$

holds. In other words, compute the amplification factor, $g(\xi)$, and show that $|g(\xi)| \le 1$, for all values of ξ, provided $\mu \le 1$.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION
FRIDAY, AUGUST 17, 2012

NAME: ______________________________

1. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be symmetric. Suppose the spectrum, denoted $\sigma(\mathsf{A}) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$, has the following ordering

$$|\lambda_1| \geq \cdots \geq |\lambda_{r-1}| > \lambda_r > |\lambda_{r+1}| \geq \cdots \geq |\lambda_n| \geq 0.$$

Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an orthonormal basis of eigenvectors of A, with $\mathsf{A}\mathbf{x}_k = \lambda_k \mathbf{x}_k$, for $k = 1, \ldots, n$. The inverse iteration is as follows: given the real number $\theta \notin \sigma(\mathsf{A})$ – which is closer to λ_r than any other element of $\sigma(\mathsf{A})$ – and $\mathbf{v}^{(0)}$, with $\left\|\mathbf{v}^{(0)}\right\|_2 = 1$ and $\mathbf{x}_r^T \mathbf{v}^{(0)} > 0$, define

$$\mathbf{v}^{(m+1)} := \frac{(\mathsf{A} - \theta \mathsf{I})^{-1} \mathbf{v}^{(m)}}{\left\|(\mathsf{A} - \theta \mathsf{I})^{-1} \mathbf{v}^{(m)}\right\|_2}.$$

Prove that $\mathbf{v}^{(m)} \to \mathbf{x}_r$, as $m \to \infty$.

2. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$. Define

$$\mathsf{S}_n := \mathsf{I} + \mathsf{A} + \cdots + \mathsf{A}^n .$$

   (a) Prove that the sequence $\{\mathsf{S}_n\}_{n=0}^{\infty}$ converges if and only if A is convergent.

   (b) Prove that if A is convergent, then $\mathsf{I} - \mathsf{A}$ is non-singular and

$$\lim_{n\to\infty} \mathsf{S}_n = (\mathsf{I} - \mathsf{A})^{-1} .$$

3. Suppose the Conjugate Gradient algorithm is applied to solve $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathsf{A} \in \mathbb{R}^{n\times n}$ is SPD and $\mathbf{0} \neq \mathbf{b} \in \mathbb{R}^n$, using $\mathbf{x}_0 = \mathbf{0}$. Prove that, if the iteration has not already converged ($\mathbf{r}_{i-1} \neq \mathbf{0}$), then there is a unique polynomial $p_i \in \mathcal{P}_i$ that minimizes $\|p(\mathsf{A})\mathbf{e}_0\|_{\mathsf{A}}$. Show that the iterate $\mathbf{x}_i$ has the error $\mathbf{e}_i = p_i(\mathsf{A})\mathbf{e}_0$ and, consequently,

$$\frac{\|\mathbf{e}_i\|_{\mathsf{A}}}{\|\mathbf{e}_0\|_{\mathsf{A}}} \leq \inf_{p\in\mathcal{P}_i} \max_{\lambda\in\sigma(\mathsf{A})} |p(\lambda)| .$$

4. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be invertible and $\mathbf{b} \in \mathbb{C}^n$. Prove that the classical Jacobi iteration method for approximating the solution to $\mathsf{A}\mathbf{x} = \mathbf{b}$ is convergent, for any starting value $\mathbf{x}_0$, if A is strictly diagonally dominant, *i.e.*,

$$|a_{i,i}| > \sum_{k\neq i} |a_{i,k}| , \quad \forall\, i = 1, \ldots, n.$$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$, and, for some $\xi \in \mathbb{R}$, $f(\xi) = 0$, but $f'(\xi) \neq 0$. Assume that, for some $\delta > 0$, $f \in C^1(I_\delta)$, where $I_\delta = [\xi - \delta, \xi + \delta]$. Prove that the sequence $\{x_k\}$ defined by the secant method,

$$x_{k+1} = x_k - \frac{f(x_k)}{s_k}, \quad s_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}},$$

converges (at least) linearly to the root ξ as $k \to \infty$, provided x_{-1} and x_0 are sufficiently close to ξ.

6. The implicit midpoint method for solving the IVP $y'(x) = f(x, y(x))$, $x \in [a, b]$, $y(x_0) = y_0$, $x_0 \in [a, b)$, is defined as

$$\eta_{i+1} = \eta_i + hf\left(x_i + \frac{h}{2}, \frac{\eta_i + \eta_{i+1}}{2}\right), \quad i = 0, 1, 2, \ldots,$$

with $\eta_0 = y_0$, $x_i = x_0 + ih$, $h > 0$. If $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz in its second variable, prove that the method is globally convergent, and the global rate of convergence is second order, assuming that $y \in C^3[a, b]$. (You may assume that the local truncation error is second-order without proof.)

7. Consider the trapezoidal method for solving the IVP $y'(x) = f(x, y(x))$, $x \in [a, b]$, $y(x_0) = y_0$, $x_0 \in [a, b)$:

$$\eta_{n+1} = \eta_n + \frac{h}{2}\Big[f(x_n, \eta_n) + f(x_{n+1}, \eta_{n+1})\Big].$$

with $\eta_0 = y_0$, $x_i = x_0 + ih$, $h > 0$.

   (a) Prove that the local truncation error is second order if $y \in C^3[a, b]$.

   (b) Prove that the method is A-stable.

8. Let V be a Hilbert space with inner product $(\cdot, \cdot)_V$ and norm $\|v\|_V := \sqrt{(v, v)_V}$, $\forall\, v \in V$. Suppose $a : V \times V \to \mathbb{R}$ is a symmetric bilinear form that is *continuous*, *i.e.*, $|a(u, v)| \leq \gamma \|u\|_V \|v\|_V$, $\exists\, \gamma > 0$, $\forall\, u, v \in V$, and *coercive*, *i.e*, $\alpha \|u\|_V^2 \leq |a(u, u)|$, $\exists\, \alpha > 0$, $\forall\, u \in V$. Suppose $L : V \to \mathbb{R}$ is linear and *bounded*, *i.e.*, $|L(u)| \leq \lambda \|u\|_V$, for some $\lambda > 0$, $\forall\, u \in V$. Let u satisfy $a(u, v) = L(v)$, for all $v \in V$.

   (a) Galerkin approximation: Suppose that $S_h \subset V$ is finite dimensional. Prove that there exists a unique $u_h \in V$ that satisfies $a(u_h, v) = L(v)$, for all $v \in S_h$. (Hint: show the stiffness matrix is SPD.)

   (b) Prove that the Galerkin approximation is stable: $\|u_h\|_V \leq \frac{\lambda}{\alpha}$.

   (c) Prove Cea's lemma:

$$\|u - u_h\|_V \leq \frac{\gamma}{\alpha} \inf_{w \in S_h} \|u - w\|_V.$$

9. Consider the Lax Friedrichs approximation scheme,

$$w_\ell^{n+1} = \frac{1}{2}\left(w_{\ell-1}^n + w_{\ell+1}^n\right) - \frac{\mu}{2}\left(w_{\ell+1}^n - w_{\ell-1}^n\right), \quad \mu = \frac{a\,s}{h},$$

for the Cauchy problem for the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, where $a > 0$.

(a) Prove that, if $s = C_1 h$, where $C_1 > 0$, the local truncation error satisfies the estimate

$$|T_\ell^n| \leq C_0\,(s+h)\ ,$$

where $C_0 > 0$ is a constant independent of s, and h.

(b) Use the von Neumann stability analysis to show that the Lax-Friedrichs scheme is stable provided the CFL condition

$$0 < \mu = \frac{a\,s}{h} \leq 1$$

holds.

10. Consider the linear reaction-diffusion problem

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad &\text{for} \quad 0 < x < 1\,, \quad 0 < t \leq T\,, \\ u(0,t) = 0 = u(1,t) \quad &\text{for} \quad 0 \leq t \leq T\,, \\ u(x,0) = g(x) \quad &\text{for} \quad 0 \leq x \leq 1\,. \end{aligned}$$

The 'Crank-Nicolson' scheme for this problem is written as

$$w_\ell^{n+1} = w_\ell^n + \frac{\mu}{2}\left(w_{\ell-1}^{n+1} - 2w_\ell^{n+1} + w_{\ell+1}^{n+1} + w_{\ell-1}^n - 2w_\ell^n + w_{\ell+1}^n\right) - \frac{s}{2}\left(w_\ell^{n+1} + w_\ell^n\right),$$

where $\mu = \frac{s^2}{h}$. Prove that the method is stable in the sense that

$$\left\|\mathbf{w}^{n+1}\right\|_\infty \leq \left\|\mathbf{w}^n\right\|_\infty,$$

for all $n \geq 0$, if $0 < \mu + \frac{s}{2} \leq 1$.

Numerical Mathematics Preliminary Examination
Friday January 6, 2012

Name: ______________________________

I. Numerical Linear Algebra

1. Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite (HPD), represented as

$$A = \begin{bmatrix} \alpha & \mathbf{p}^H \\ \mathbf{p} & \hat{A} \end{bmatrix},$$

where α is a scalar, $\mathbf{p} \in \mathbb{C}^{n-1}$, and $\hat{A} \in \mathbb{C}^{(n-1)\times(n-1)}$. After 1 step of Gaussian elimination (without pivoting), A will be reduced to the matrix

$$\begin{bmatrix} \alpha & \mathbf{p}^H \\ \mathbf{0} & B \end{bmatrix},$$

where $B \in \mathbb{C}^{(n-1)\times(n-1)}$. Prove that B is HPD. In doing so, also prove that the corresponding diagonal elements of B are smaller than those of $\hat{A}$.

2. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD). Suppose $P \in \mathbb{R}^{n\times m}$, $m \le n$, is full rank.

   (a) Show that $A_C := P^TAP$ is invertible.

   (b) Define $Q_A := PA_C^{-1}P^TA$. Show that Q_Au is the best approximation of u in Range(P) with respect to the A-inner product, which is defined as follows:

$$(u,v)_A = (Au,v) \qquad \forall\, u,v \in \mathbb{R}^n,$$

   where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$.

3. Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$. Suppose x and $\hat{x}$ solve $Ax = b$ and $(A+\delta A)\hat{x} = b + \delta b$, respectively. Assuming that $\|A^{-1}\|_2\,\|\delta A\|_2 < 1$, show that

$$\frac{\|\delta x\|_2}{\|x\|_2} \le \frac{\kappa_2(A)}{1-\kappa_2(A)\frac{\|\delta A\|_2}{\|A\|_2}}\left(\frac{\|\delta A\|_2}{\|A\|_2} + \frac{\|\delta b\|_2}{\|b\|_2}\right),$$

where $\delta x := \hat{x} - x$.

4. Let $A \in \mathbb{R}^{m\times m}$ be symmetric, and suppose λ_i is an eigenvalue with corresponding eigenvector x_i. Given $x \in \mathbb{R}^m$, with $\|x - x_i\|_2 = O(\epsilon)$, $1 \gg \epsilon > 0$, show that

$$\left|\frac{x^TAx}{x^Tx} - \lambda_i\right| = O(\epsilon^2).$$

II. Numerical Solutions of Nonlinear Equations

5. Let $\{x^{(n)}\}$ be a sequence generated by Newton's method. Suppose that the initial guess $x^{(0)}$ is well-chosen so that this sequence converges to the exact solution x_*. Prove that if $f(x_*) = f'(x_*) = \cdots = f^{(m-1)}(x_*) = 0$, $f^{(m)}(x_*) \neq 0$, $x^{(n)}$ converges q-linearly to x_* with $\lim_{k\to\infty} \dfrac{e^{(k+1)}}{e^{(k)}} = \dfrac{m-1}{m}$.

III. Numerical Solutions of ODEs

6. Show that, if z is a non-zero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(x_{n+2}, y_{n+2}),$$

then the real part of z must be positive. Thus deduce that this method is A-stable.

7. Consider the scheme

$$y_{n+3} + \alpha(y_{n+2} - y_{n+1}) - y_n = h\beta(f_{n+2} + f_{n+1})$$

for approximating the solution to

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0.$$

(a) Find the range of α and β, such that the resulting three-step method is stable;

(b) Find the value of α and β, such that this method has order of accuracy 4;

(c) Can one adjust α and β to obtain a convergent fourth-order method? Justify your answer.

IV. Numerical Solutions of PDEs

8. Consider the one dimensional heat problem

$$\begin{aligned} u_t - u_{xx} &= 0, \qquad 0 < x < 1, \quad t > 0 \\ u(t,0) &= u(t,1) = 0, \qquad t \geq 0, \\ u(0,x) &= u_0(x), \qquad 0 < x < 1. \end{aligned}$$

and the Crank-Nicholson method ($\mu = k/h^2$)

$$u_j^{n+1} - u_j^n = \frac{\mu}{2}\left[\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + \left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)\right].$$

Define the discrete energy as $E^n = \left(h\sum_{j=0}^{J+1}\left(u_j^n\right)^2\right)^{1/2}$. Prove the energy stability of this method by showing that $E^n \leq E^0$.

9. Suppose $a > 0$ and consider the following skewed leapfrog method for solving the advection equation $u_t + au_x = 0$:

$$u_j^{n+1} = u_{j-2}^{n-1} - (ak/h - 1)(u_j^n - u_{j-2}^n).$$

(a) What is the order of accuracy of this method?

(b) For what range of Courant number ak/h does this method satisfy the CFL condition? (Recall the CFL condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as k and h go to zero.)

(c) Show that the method is in fact stable for this range of Courant numbers by doing a von Neumann analysis.

Numerical Mathematics Preliminary Examination
August 8, 2011

Name: ______________________________

I. Numerical Linear Algebra

1. Let A be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ be QR and Cholesky factorizations of A and A^*A, respectively, with the usual normalizations r_{jj}, $u_{jj} > 0$. Is it true or false that $R = U$? Justify your answer.

2. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Prove that

$$\min\left\{\frac{\|B\|_2}{\|A\|_2} : A + B \text{ singular}\right\} = \frac{1}{\kappa_2(A)},$$

where $\kappa_2(A) := \|A\|_2\|A^{-1}\|_2$ denotes the 2-norm condition number of A.

3. Let $A, B \in \mathbb{C}^{m\times m}$, with A being a nonsingular matrix. Consider solving the linear system

$$Ax + By = b_1, \qquad Bx + Ay = b_2,$$

for the unknowns x and y.

   (a) Show that $\rho(A^{-1}B) < 1$ is a necessary and sufficient condition for convergence of the iteration scheme:

$$Ax^{k+1} = b_1 - By^k, \qquad Ay^{k+1} = b_2 - Bx^k,$$

   with an arbitrary initial guess.

   (b) If we consider a slightly modified iteration scheme:

$$Ax^{k+1} = b_1 - By^k, \qquad Ay^{k+1} = b_2 - Bx^{k+1},$$

   does the conclusion of part (a) still hold? Why yes or why no?

   (c) Under the assumption that both iterative methods converge, determine which method converges faster? Justify your answer.

4. Let $A \in \mathbb{R}^{n\times n}$ be a SPD matrix. Any two vectors $u, v \in \mathbb{R}^n$ are called A-orthogonal if $v^TAu = 0$. Prove that every subspace of $\mathbb{R}^n$ has an A-orthogonal basis.

II. Numerical Solutions of Nonlinear Equations

5. Show that if f is a real function of one real variable, f'' is Lipschitz continuous, and $f(x_*) = f'(x_*) = 0$ but $f''(x_*) \neq 0$ then the iteration

$$x_{n+1} = x_n - 2f(x_n)/f'(x_n),$$

converges quadratically to x_* provided x_0 is sufficiently near to x_*, but not equal to x_*.

III. Numerical Solutions of ODEs

6. Consider the two-stage Runge-Kutta method

$$y^* = y_i + ahf(y_i), \qquad y_{i+1} = y_i + h(bf(y_i) + cf(y^*)).$$

   (a) Find the relation among the coefficients a, b and c, so that the method is of order 2. Write the resulting Runge-Kutta method in the tableaux form.

   (b) Find the linear stability region of this method.

   (c) Consider the case when $a = 1/2$, $b = 0$ and $c = 1$. Prove explicitly that this method converges when applied to the IVP $y' = f(y)$, $y_0 = 1$, where f is a Lipschitz function with Lipschitz constant λ.

7. a) Find the range of $a \in \mathbb{R}$ for which the method

$$y_{n+2} + (a-1)y_{n+1} - ay_n = \frac{h}{4}\left((a+3)f(t_{n+2}, y_{n+2}) + (3a+1)f(t_n, y_n)\right),$$

is consistent and stable.

b) Apply the method with $a = -1$ to the scalar IVP $y' = y$, $y(0) = 1$ and solve exactly the resulting difference equation, considering the starting values to be $y_0 = y_1 = 1$. Show theoretically that the numerical solution does not converge as $h \to 0$ and $n \to \infty$.

IV. Numerical Solutions of PDEs

8. Consider the PDE

$$u_t = u_{xx} - \gamma u,$$

which models a diffusion with decay, provided $\gamma > 0$. Consider the numerical methods of the form

$$u_j^{n+1} = u_j^n + \frac{\mu}{2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n + u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right] - k\gamma\left[(1-\theta)u_j^n + \theta u_j^{n+1}\right],$$

where $\mu = k/h^2$ and θ is a parameter.

   (a) By computing the local truncation error, show that this method is $O(k^p + h^2)$ accurate, where $p = 2$ if $\theta = 1/2$ and $p = 1$ otherwise.

   (b) Using von Neumann analysis, show that this method is unconditionally stable if $\theta \geq 1/2$.

   (c) Show that if $\theta = 0$ then the method is stable provided $k \leq 2/\gamma$, independent of h.

9. Consider the constant coefficient advection equation in $\mathbb{R}^2$:

$$u_t + au_x + bu_y = 0, \qquad u(x, y, 0) = g(x, y),$$

where $a > 0$ and $b > 0$, and a Cartesian grid defined by the grid points $x_i = i\Delta x$, $y_j = j\Delta y$ with

$$u_{i,j}^n \approx u(x_i, y_j, t^n).$$

Furthermore, consider the following finite difference numerical method:

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{a\Delta t}{2\Delta x}(u_{i+1,j}^{n} - u_{i-1,j}^{n}) - \frac{b\Delta t}{2\Delta y}(u_{i,j+1}^{n} - u_{i,j-1}^{n}).$$

(a) Derive the local truncation error of the provided method.

(b) Is this method convergent? Justify your answer. If yes, what is the required condition to make it convergent? If no, can you modify it so that the resulting method is convergent (no need to prove)?

Numerical Mathematics Preliminary Examination
January 7, 2011

Name: ______________________________

I. Numerical Linear Algebra

1. Consider a linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$. Richardson's method is an iteration method
$$M\mathbf{x}^{k+1} = N\mathbf{x}^k + \mathbf{b}$$
with $M = \frac{1}{\omega}I$, $N = M - A = \frac{1}{\omega}I - A$, where ω is a damping factor chosen to make M approximate A as well as possible. Suppose A is positive definite and $\omega > 0$. Let λ_1 and λ_n denote the smallest and largest eigenvalues of A.
   (a) Prove that Richardson's method converges if and only if $\omega < \frac{2}{\lambda_n}$.
   (b) Prove that the optimal value of ω is $\omega_o = \frac{2}{\lambda_1+\lambda_n}$.
2. Let $A \in \mathbb{C}^{m\times n}$ $(m \geq n)$ and $\mathbf{b} \in \mathbb{C}^m$. Prove that the vector $\mathbf{x} \in \mathbb{C}^n$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{r} \perp \text{range}(A)$, where $\mathbf{r} := \mathbf{b} - A\mathbf{x}$.
3. Suppose that A, $B \in \mathbb{C}^{m\times m}$ and A is non-singular and B is singular. Prove that
$$\frac{1}{\kappa(A)} \leq \frac{\|A-B\|}{\|A\|},$$
where $\kappa(A) = \|A\| \cdot \|A^{-1}\|$, and $\|\cdot\|$ is an induced matrix norm.

II. Numerical Solutions of Nonlinear Equations

4. Let $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be twice continuously differentiable. Suppose $\mathbf{x}^* \in \Omega$ is a solution of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, and the Jacobian matrix of $\mathbf{f}$, denoted $J_{\mathbf{f}}$, is invertible at $\mathbf{x}^*$.
   (a) Prove that if $\mathbf{x}^0 \in \Omega$ is sufficiently close to $\mathbf{x}^*$, then the following iteration converges to $\mathbf{x}^*$:
$$\mathbf{x}^{k+1} = \mathbf{x}^k - J_{\mathbf{f}}\left(\mathbf{x}^0\right)^{-1}\mathbf{f}(\mathbf{x}^k).$$
   (b) Prove that the convergence is typically only linear.

III. Numerical Solutions of ODEs

5. Consider
$$y'(t) = f(t, y(t)), \quad t \geq t_0, \quad y(t_0) = y_0,$$
where $f : [t_0, t^*] \times \mathbb{R} \to \mathbb{R}$ is continuous in its first variable and Lipschitz continuous in its second variable. Prove that Euler's method converges.
6. Consider the scheme
$$y_{n+2} + y_{n+1} - 2y_n = h\left(f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right)$$

for approximating the solution to

$$y'(t) = f(t, y(t)) , \quad t \ge t_0 , \quad y(t_0) = y_0 .$$

What is the order of the scheme? Is it a convergent scheme? Is it A-stable? Justify your answers.

IV. Numerical Solutions of PDEs

7. Consider the Crank-Nicholson scheme applied to the diffusion equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $t > 0$, $-\infty < x < \infty$.
   (a) Show that the amplification factor in the Von Neumann (Fourier) analysis of the scheme is

$$g(\xi) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z} , \quad z = 2\frac{\Delta t}{\Delta x^2} \left(\cos\left(\Delta x \xi\right) - 1\right) .$$

   (b) Use the result of part (a) to show that the scheme is stable.

8. Consider the explicit scheme

$$u_\ell^{n+1} = u_\ell^n + \mu\left(u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n\right) - \frac{b\mu\Delta x}{2}\left(u_{\ell+1}^n - u_{\ell-1}^n\right) , \quad \begin{matrix} 0 \le n \le N \\ 1 \le \ell \le L \end{matrix} ,$$

for the convection-diffusion problem

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x} \quad & \text{for} \quad 0 \le x \le 1 , \quad 0 \le t \le t^\star , \\ u(0,t) = u(1,t) = 0 \quad & \text{for} \quad 0 \le t \le t^\star , \\ u(x,0) = g(x) \quad & \text{for} \quad 0 \le x \le 1 , \end{aligned}$$

where $b > 0$, $\mu = \frac{\Delta t}{(\Delta x)^2}$, $\Delta x = \frac{1}{L+1}$, and $\Delta t = \frac{t^\star}{N}$. Prove that, under suitable restrictions on μ and Δx, the error grid functions $\mathbf{e}^n$ satisfy the estimate

$$\|\mathbf{e}^n\|_\infty \le t^\star C\left(\Delta t + (\Delta x)^2\right) ,$$

for all n such that $n\Delta t \le t^\star$, where $C > 0$ is a constant.

NUMERICAL MATHEMATICS PRELIMINARY EXAMINATION
AUGUST 11, 2010

NAME: ______________________________

I. Numerical Linear Algebra

1. Let $A \in \mathbb{C}^{m\times n}$ $(m \geq n)$ and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.
   (a) Prove that A has rank n if and only if all the diagonal entries of $\hat{R}$ are non-zero.
   (b) Suppose $\text{rank}(A) = n$, and define $P = \hat{Q}\hat{Q}^*$. Prove that $\text{range}(P) = \text{range}(A)$.
   (c) What type of matrix is P?

2. Suppose that $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite (SPD).
   (a) Show that $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A\mathbf{x}}$ defines a vector norm.
   (b) Let the eigenvalues of A be ordered so that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Show that
   $$\sqrt{\lambda_1}\,\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_A \leq \sqrt{\lambda_m}\,\|\mathbf{x}\|_2$$
   for any $\mathbf{x} \in \mathbb{R}^m$.
   (c) Let $\mathbf{b} \in \mathbb{R}^m$ be given. Prove that $\mathbf{x}_* \in \mathbb{R}^m$ solves $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}_*$ minimizes the quadratic function $f : \mathbb{R}^m \to \mathbb{R}$ defined by
   $$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}\ .$$

3. Suppose that $A \in \mathbb{R}^{m\times m}$ is SPD and $\mathbf{b} \in \mathbb{R}^m$ is given. The n^{th} Krylov subspace is defined as $\mathcal{K}_n := \langle \mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{n-1}\mathbf{b}\rangle$. Let $\{\mathbf{x}_j\}_{j=0}^{n-1}$, $\mathbf{x}_0 = \mathbf{0}$, denote the sequence of vectors generated by the conjugate gradient algorithm. Prove that if the method has not already converged after $n-1$ iterations, *i.e*, $\mathbf{r}_{n-1} := \mathbf{b} - A\mathbf{x}_{n-1} \neq \mathbf{0}$, then the n^{th} iterate $\mathbf{x}_n$ is the unique vector in $\mathcal{K}_n$ that minimizes $\phi(\mathbf{y}) := \|\mathbf{x}_* - \mathbf{y}\|_A^2$, where $\mathbf{x}_* := A^{-1}\mathbf{b}$.

4. Prove that $A \in \mathbb{R}^{m\times m}$ is SPD if and only if it has a Cholesky factorization.

II. Numerical Solutions of Nonlinear Equations

5. Assume that $f : \mathbb{R} \to \mathbb{R}$, $f \in C^2(\mathbb{R})$, $f'(x) > 0$, for all $x \in \mathbb{R}$, and $f''(x) > 0$, for all $x \in \mathbb{R}$.
   (a) Suppose that a root $\xi \in \mathbb{R}$ exists. Prove that it is unique. Exhibit a function satisfying the assumptions above that has no root.
   (b) Prove that for any starting guess $x_0 \in \mathbb{R}$, Newton's method converges, and the convergence rate is quadratic.

III. Numerical Solutions of ODEs

6. Determine all the values of θ for which the θ-method,

$$y_{n+1} = y_n + h\Big[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\Big] ,$$

is A-stable.

7. Show that the explicit multistep method

$$y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h[\beta_2 f(t_{n+2}, y_{n+2}) + \beta_1 f(t_{n+1}, y_{n+1}) + \beta_0 f(t_n, y_n)]$$

for approximating the solution to the initial value problem

$$y'(t) = f(t, y(t)) , \quad y(t_0) = y_0$$

is fourth order only if $\alpha_0 + \alpha_2 = 8$ and $\alpha_1 = -9$. Prove that this method cannot be both fourth order and convergent.

IV. Numerical Solutions of PDEs

8. Consider the Crank-Nicolson scheme

$$u_\ell^{n+1} = u_\ell^n + \frac{\mu}{2}\left(u_{\ell-1}^{n+1} - 2u_\ell^{n+1} + u_{\ell+1}^{n+1} + u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n\right)$$

for approximating the solution to the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on the intervals $0 \le x \le 1$ and $0 < t \le t^*$ with the boundary conditions $u(0,t) = u(1,t) = 0$.

(a) Show that the scheme may be written in the form $\mathbf{u}^{n+1} = A\mathbf{u}^n$, where $A \in \mathbb{R}^{m\times m}_{\text{sym}}$ (the space of $m \times m$ sysmmetric matrices) and

$$\|A\mathbf{x}\|_2 \le \|\mathbf{x}\|_2 ,$$

for any $\mathbf{x} \in \mathbb{R}^m$, regardless of the value of μ.

(b) Show that

$$\|A\mathbf{x}\|_\infty \le \|\mathbf{x}\|_\infty ,$$

for any $\mathbf{x} \in \mathbb{R}^m$, provided $\mu \le 1$. (In other words, the scheme may only be conditionally stable in the max norm.)

9. Consider the Lax-Wendroff scheme,

$$u_\ell^{n+1} = u_\ell^n + \frac{a^2(\Delta t)^2}{2(\Delta x)^2}\left(u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{\ell+1}^n - u_{\ell-1}^n\right) ,$$

for the approximating the solution of the Cauchy problem for the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, where $a > 0$. Use Von Neumann's method to show that the Lax-Wendroff scheme is stable provided the CFL condition

$$\frac{a\Delta t}{\Delta x} \le 1$$

is enforced.

Numerical Mathematics Preliminary Exam

January 8, 2010

I. Numerical Linear Algebra

1. Let $\mathbf{x} \in \mathbf{R}^m$ and $\mathbf{y} \in \mathbf{R}^n$. A trivial algorithm for computing the rank-one matrix (also called the outer product of $\mathbf{x}$ and $\mathbf{y}$) $A = \mathbf{x}\mathbf{y}^T$ is to compute the mn products $x_i y_j$ with $\otimes$ and collect them into a matrix $\widetilde{A}$.

   (a) Determine whether this algorithm is *stable*. Justify your answer.

   (b) Determine whether this algorithm is *backward stable*. Justify your answer.

2. Suppose $A \in \mathbf{R}^{m\times n}$ $(m \geq n)$ has full rank.

   (a) Prove that $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.

   (b) Is the assertion of (a) still true if the A is rank-deficient? Why yes or why no?

3. Let A be a symmetric matrix and $\mathbf{x}$ an approximation to an eigenvector $\mathbf{v}$ of A, with $A\mathbf{v} = \lambda\mathbf{v}$ for some λ. Let μ denote the Rayleigh quotient for $\mathbf{x}$. Show that if $\|\mathbf{x}-\mathbf{v}\| = \epsilon$ then we have $|\mu - \lambda| = O(\epsilon^2)$.

4. Consider the matrices of the following form:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \qquad \rho \in \mathbf{R}, \quad |\rho| < 1.$$

   (a) Determine the values of ρ such that the Jacobi iterative method converges for arbitrary starting value $\mathbf{x}_0 \in \mathbf{R}^2$.

   (b) Determine the values of ρ such that the Gauss-Seidel iterative method converges for arbitrary starting value $\mathbf{x}_0 \in \mathbf{R}^2$.

II. Solutions of Nonlinear Equations

5. Let $f : \mathbf{R}^n \to \mathbf{R}$ be twice continuously differentiable. Suppose $\mathbf{x}^*$ is an isolated root of f and the Jacobian of f at $\mathbf{x}^*$ $(J(\mathbf{x}^*))$ is non-singular. Determine condition(s) on ϵ so that if $\|\mathbf{x}_0 - \mathbf{x}^*\|_2 < \epsilon$ then the following iteration converges to $\mathbf{x}^*$:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J(\mathbf{x}_0)^{-1} f(\mathbf{x}_k), \qquad k = 0, 1, 2, \ldots$$

III. Numerical Solutions of ODEs

6. Let D and Q be $n \times n$ matrices, with Q invertible and $Q^{-1}DQ = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Show that applying any Runge-Kutta method to solving the system

$$y' = Dy$$

   is equivalent to applying the same Runge-Kutta method to solving the system

$$z' = \Lambda z,$$

   where $z = Q^{-1}y$.

7. Consider the *centered difference method*

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n)$$

for the initial value problem (IVP) $y' = f(t, y)$, $y(0) = y_0$.

(a) Let $f(t, y) = \cos y$. Prove (by the definition) that the *centered difference method* converges provided that y_0 and y_1 converges (as $h \to 0^+$) to $y(0)$ and $y(t_1)$, respectively.

(b) Assume that $y(0) - y_0 = O(h^4)$ and $y(t_1) - y_1 = O(h^4)$, determine the order of convergence for the *centered difference method.*

IV. Numerical Solutions of PDEs

8. Consider the following two-step method for solving the 1-D heat equation $u_t - u_{xx} = 0$:

$$\frac{U_i^{n+2} - U_i^n}{2\Delta t} - \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2} = 0.$$

(a) Determine the order of the truncation error of this method (in both space and time).

(b) Use the Fourier Analysis to determine for what value $\mu = \frac{\Delta t}{h^2} > 0$ (if any) that the method is stable.

(c) Is this method convergent for $0 < \mu < \frac{1}{4}$?