Prelim PDEs — August 2023

Problem 1:

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $f \in C^0(\mathbb{R} \to \mathbb{R})$ strictly increasing. Consider the boundary value problem

$$\begin{aligned} &\Delta u = f(u) \quad \text{in } \Omega \\ &u(x) + a(x)\partial_\nu u(x) = g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{$*$}$$

where a and g are continuous functions on $\bar{\Omega}$ and $a > 0$. Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution.

(a) Show: The solution of the BVP (*) is unique.

(b) Assuming F to be an antiderivative of f, show: u solves the BVP if and only if u minimizes the functional

$$I[u] := \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u))\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{1}{a(x)}(u(x) - g(x))^2\,dS(x)$$

among all $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying the BC.

Problem 2:

Consider the problem

$$\begin{aligned} &u_t + uu_x = u - \tfrac{1}{4}x \\ &u(x,0) = g(x)\ . \end{aligned}$$

(a) Write a formula for the characteristic curves $(t(\tau), x(\tau), z(\tau))$.

(b) Characterize all functions g that give rise to a *global* classical solution (i.e., $u \in C^1(\mathbb{R} \times \mathbb{R})$.)

Problem 3:

Prove: If e^u is harmonic in $\mathbb{R}^n$, then u is constant.

Problem 4:

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. *For this problem, we may use the following version of the weak maximum principle without proof:*

Suppose that $T > 0$, and $u \in C^2(\overline{\Omega} \times [0,T])$ is a solution to

$$\begin{cases} u_t - \Delta u + c(x,t)u \le 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) \le 0, & x \in \Omega, \end{cases}$$

where for $c_0 > 0$, $c(x,t) \ge -c_0$, and ν is the outward unit normal to $\partial\Omega$. Then $u \le 0$ in $\bar{\Omega} \times [0,T]$.

Suppose that $u \in C^2(\overline{\Omega} \times [0, \infty))$ is solution to the initial-Neumann problem

$$\begin{cases} u_t - \Delta u = f(u) \quad \text{in } \Omega \times (0, \infty) \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega \times (0, \infty), \\ u(x, 0) = g(x), \qquad x \in \Omega \end{cases}$$

where $f(u) = u(1-u)(1+u)$ and $g \in C^0(\overline{\Omega})$. For a given constant v_0, denote by $v(t; v_0)$ the solution to the initial value problem

$$\begin{cases} \dfrac{dv}{dt} = f(v) \\ v(0) = v_0, \end{cases} \tag{ODE}$$

(a) Show that

$$v(t; m) \leq u(x, t) \leq v(t; M), \qquad \forall (x, t) \in \overline{\Omega} \times [0, \infty)$$

where $m = \min_{\overline{\Omega}} g$ and $M = \max_{\overline{\Omega}} g$

(b) Show that if $g(x) > 0$, for all $x \in \overline{\Omega}$, then $\lim_{t\to\infty} u(x, t) = 1$ uniformly for $x \in \overline{\Omega}$. [Hint: What can you say about the behavior of the solution of (ODE) if $v_0 > 0$?]

Problem 5:

Consider the following 1d diffusion equation with a nonlinear term

$$u_t - bu_{xx} + a(u_x)^2 = 0 \qquad b > 0, \text{ and } a \neq 0 \text{ constant.} \tag{$*$}$$

(a) Show that the transformation $v(x, t) = e^{-\frac{a}{b}u(x,t)}$ transforms the nonlinear equation $(*)$ into

$$v_t - bv_{xx} = 0.$$

(b) Apply part (a) to find an explicit formula for a solution of the initial value problem

$$\begin{cases} u_t - bu_{xx} + a(u_x)^2 = 0, \quad t > 0, x \in \mathbb{R} \qquad (\text{for } b > 0 \text{ and } a \neq 0) \\ u(x, 0) = g(x), \end{cases}$$

Give a condition on the solution u that implies its uniqueness.

Question 6:

For $k = 1, 2$ let φ_k, ψ_k be smooth compactly supported functions defined on $\mathbb{R}$, and assume that u_k is the solution to the wave equation

$$u_{tt} - a^2 u_{xx} = f \quad \text{in} \quad \mathbb{R} \times (0, \infty)$$

that satisfies

$$u(x,0) = \varphi_k(x) \quad \text{and} \quad u_t(x,0) = \psi_k(x) \quad \text{for} \quad x \in \mathbb{R}$$

where $a > 0$ is a fixed number and $f : \mathbb{R} \times [0, \infty)$ is a given smooth function. Prove that for every $\varepsilon > 0$ and $T > 0$, there is $\delta > 0$ such that if

$$\sup_{x\in\mathbb{R}} |\varphi_1(x) - \varphi_2(x)| \le \delta \quad \text{and} \quad \left(\int_{-\infty}^{\infty} |\psi_1(x) - \psi_2(x)|^2 dx \right)^{1/2} \le \delta$$

then

$$\sup_{x\in\mathbb{R}, t\in[0,T]} |u_1(x,t) - u_2(x,t)| \le \varepsilon.$$

Question 7:
Let $c : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a function such that $\varphi = 0$ on B_1. Assume that u is a smooth solution of the nonlinear wave equation

$$u_{tt} - \Delta u + c(x,t)|\nabla u|^2 + u^3 = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty)$$

that satisfies the initial data

$$u(x,0) = \varphi(x), \quad u_t(x,0) = 0 \quad x \in \mathbb{R}^3.$$

Prove that $u = 0$ in the cone

$$K = \{(x,t) \in \mathbb{R} \times [0,\infty) : 0 \le t \le 1, |x| \le 1 - t\}.$$

Here B_ρ is the ball in $\mathbb{R}^3$ with radius $\rho > 0$ and centered at the origin.

PDE Preliminary Exam, January 2022

There are 7 problems in this exam. Do all of them.

1. Suppose that $f(x)$ is smooth and nonnegative

$$u_t + xu_x = -u^2, \quad (x,t) \in \mathbb{R} \times (0,\infty)$$
$$u(x,0) = f(x)$$

   (a) Write a formula for the solution u and discuss the behavior of $u(x,t)$ as $t \to \infty$.

   (b) If $f(x) > 0$ on $0 < x < 1$ and $f(x) = 0$ elsewhere, plot the region in the (x,t)-plane where the (weak) solution $u(x,t) > 0$.

2. For $r > 0$, let $B_r = B_r(0) \subset \mathbb{R}^n$. Suppose that $u \in C^2(B_1) \cap C(\overline{B}_1)$ such that $\Delta u \geq 0$ in B_1. For $\epsilon > 0$, $x_0 \in \partial B_1$ and $\alpha \geq 2n+1$ let

$$h_\epsilon(x) = u(x) - u(x_0) + \epsilon\big(e^{-\alpha|x|^2} - e^{-\alpha}\big), \quad x \in \overline{B}_1$$

   (a) Let $D = B_1 \setminus B_{1/2}$ and prove that $\Delta h_\epsilon(x) > 0$ for all $x \in D$.

   (b) Suppose that $u(x) < u(x_0)$ for all $x \in \overline{B}_1 \setminus \{x_0\}$. Prove that there exists $\epsilon_0 > 0$ such that

$$\max_{x \in \overline{D}} h_\epsilon(x) = h_\epsilon(x_0), \quad \forall \epsilon \in (0, \epsilon_0),$$

   and then conclude that

$$\frac{\partial u}{\partial \nu}(x_0) \geq 2\alpha\epsilon e^{-\alpha}$$

   where ν is the outward normal vector on ∂B_1 at x_0.

3. For $B_1 = B_1(0) \subset \mathbb{R}^n$, suppose that the functions $a, f \in C(\overline{B}_1)$ and $g \in C(\partial B_1)$. Suppose also that $a(x) \geq 0$ for all $x \in B_1$. Prove that there is at most one solution $u \in C^2(B_1) \cap C(\overline{B}_1)$ of

$$\begin{cases} -a(x)\Delta u(x) + (1-|x|^2)u(x) &= f(x) \quad \text{for } x \in B_1 \\ u(x) &= g(x) \quad \text{for } x \in \partial B_1. \end{cases}$$

Note: the function a may not be differentiable in B_1.

4. Let $f \in C_0^\infty(\mathbb{R}^n)$, $b \in \mathbb{R}$ and consider

$$u_t = \Delta u - x \cdot \nabla u + (b + \frac{1}{4}|x|^2)u \quad \text{on } \mathbb{R}^n \times (0,\infty)$$
$$u = f \quad \text{on } \mathbb{R}^n \times t = 0$$

Prove that the equation has a solution $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ satisfying: for some $c, \alpha, r, t_0 > 0$, $\quad |u(x,r)| \le ce^{-\alpha|x|^2}$ holds for all $|x| > r$, $0 < t < t_0$.

(Hint: Show that $g(x,t) = e^{\frac{-1}{4}|x|^2 - (b+\frac{n}{2})t}$ solves the system $g_t - \Delta g + (b + \frac{1}{4}|x|^2)g = 0$. What IVP does $v = gu$ solve?)

5. Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega \in C^\infty$, $T > 0$. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(u) > 0$ for $u < 0$ and $u > 1$. Let also $g \in C^0(\Omega)$ with $0 \le g \le 1$ on $\overline{\Omega}$. For $\Omega_T = \Omega \times (0,T]$, suppose now that $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ is a solution of

$$u_t = \Delta u + |\nabla u|^2 - u \quad \text{on } \Omega_T,$$
$$\frac{\partial u}{\partial \nu} + f(u) = 0 \quad \text{on } \partial\Omega \times (0,T]\ ,$$
$$u(x,0) = g(x) \quad \text{for } x \in \Omega.$$

   (a) Prove that $0 \le u \le 1$ on $\overline{\Omega_T}$.

   (b) If g is nonconstant on Ω, prove that $0 < u < 1$ on Ω_T.

6. Consider the initial value problem

$$u_{tt}(x,t) - \Delta u(x,t) = q(x)e^t \quad (x) \in \mathbb{R}^3 \times \mathbb{R},$$
$$u(x,0) = 0, \quad x \in \mathbb{R}^3,$$
$$u_t(x,0) = 0, \quad x \in \mathbb{R}^3,$$

where q is smooth with $q(x) = 0$ for $|x| \ge r > 0$ for some fixed r. Show that there is a function $v(x)$ such that for each $x \in \mathbb{R}^3$,

$$u(x,t) - v(x)e^t \to 0, \quad \text{as } t \to \infty.$$

Hint: Use for a fact (without proof) $v(x) = \dfrac{1}{(4\pi)^{3/2}} \displaystyle\int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-\tau - \frac{|x-y|^2}{4\tau}}}{\tau^{3/2}} q(y)\,dy\,d\tau$ solves $-\Delta v + v(x) = q(x)$ for $x \in \mathbb{R}^3$. Prove that that for any $x \in \mathbb{R}^3$, $(1+|z|)(|v(x+z)| + |\nabla v(x+z)|) \to 0$ as $|z| \to \infty$.

7. Suppose that Ω is a bounded C^1-domain in $\mathbb{R}^n$, $f \in C(\overline{\Omega} \times [0,\infty))$, $\phi \in C^1(\overline{\Omega})$, $\psi \in C(\overline{\Omega})$ are given, and $u \in C^2(\overline{\Omega} \times [0,\infty))$ solves the initial/boundary-value problem

$$\text{(IBVP)} \qquad \begin{aligned} u_{tt} - \Delta u = f \quad & \text{in } \Omega \times (0,\infty), \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \quad & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x,t) = 0, \quad & \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

(a) Show that for any $t > 0$

$$\left(\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}\right)^{1/2} \leq \left(\|\psi\|^2_{L^2(\Omega)} + \|\nabla \phi\|^2_{L^2(\Omega)}\right)^{1/2} + \int_0^t \|f(\cdot,s)\|_{L^2(\Omega)} ds.$$

(b) Show that (IVBP) has at most one $C^2(\overline{\Omega} \times [0,\infty))$ solution.

PDE Preliminary Exam, August 2021

1. Suppose that g is a smooth function on $\mathbb{R}$ and consider the initial value problem

$$\begin{aligned} e^x u_x + u_y &= u \\ u(x,0) &= g(x). \end{aligned}$$

Write a formula for the solution. Find the domain of definition of the solution.

2. Let $B_2(0) \subset \mathbb{R}^n$, a ball centered at the origin with radius 2 and define the operator

$$Lu := \Delta u + \mathbf{b} \cdot \nabla u + (4 - |x|^2)u,$$

where $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ is a given vector of smooth functions on $\overline{B_2(0)}$. Suppose that for some $\lambda > 4$ the function $u \in C^2(\overline{B_2(0)})$ satisfies

$$\begin{aligned} Lu &= \lambda u \quad \text{in } B_2(0) \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial B_2(0). \end{aligned} \tag{1}$$

(a) Show that for large $\eta > 0$ the function $v(x) = e^{-\eta|x|^2} - e^{-4\eta}$ satisfies the inequality

$$\begin{aligned} Lv &\geq \lambda v \quad \text{in } B_2(0) \setminus B_1(0) \\ v &= 0 \quad \text{on } \partial B_2(0) \\ v &> 0 \quad \text{on } \partial B_1(0). \end{aligned}$$

(b) Prove that the solution u of (1) cannot attain its positive maximum in $B_2(0)$.

(c) Prove that the solution u of (1) can have no positive maximum in $\overline{B_2(0)}$. [*Hint: If $x_0 \in \partial B_2(0)$ such that $u(x_0) > 0$ is a maximum of u, then for appropriately chosen small ϵ work with the function $w = u + \epsilon v - u(x_0)$ on $B_2(0) \setminus B_1(0)$ where v is as in part (a).*]

(d) Conclude that the solution u of (1) is identically 0.

3. Suppose that u is harmonic on $\mathbb{R}^n$ and $B_1(0)$ represents the unit ball. For any $t > 0$ define

$$I(t) = \int_{\partial B_1(0)} u(ty)u\left(\frac{y}{t}\right) dS_y.$$

Show that I is a constant function.

4. Let α, γ be positive numbers, $\beta \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ be given. Consider the Cauchy problem

$$\begin{aligned} \alpha u_t + \mathbf{b} \cdot \nabla u + \beta u &= \gamma \Delta u \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) &= g(x), \quad \text{on } \mathbb{R}^n \end{aligned} \tag{2}$$

where g is compactly supported smooth function.

(a) Find $\kappa, \mu \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$ so that $v(x,t) = e^{\kappa t} u(\mu x + \mathbf{a}t, t)$ solves

$$\begin{aligned} v_t &= \Delta v \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ v(x,0) &= g(\mu x) \quad \text{on } \mathbb{R}^n. \end{aligned}$$

(b) Write down an explicit formula for a solution $u(x,t)$ of (2).

5. Let Ω be a bounded domain in $\mathbb{R}^n$, c be continuous in $\overline{\Omega} \times [0,T]$ with $c \geq -c_0$ for a nonnegative constant c_0, and u_0 be continuous in Ω with $u_0 \geq 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $xf(x) \leq 0$ for all $x \in \mathbb{R}$. Suppose that $u \in C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$ is a solution of

$$\begin{aligned} u_t - \Delta u + cu &= uf(u) \quad \text{in } \Omega \times (0,T] \\ u(\cdot, 0) &= u_0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T] \end{aligned}$$

Prove that

$$0 \leq u(x,t) \leq e^{c_0 T} \sup_{\Omega} u_0, \quad \text{for all } (x,t) \in \Omega \times (0,T].$$

Hint: For the lower bound work on $w = u\,e^{-Mt}$ for a suitable choice of a constant M.

6. Let Ω be a bounded smooth domain. For given smooth functions $V(x)$ and $h(x)$ in $\overline{\Omega}$, consider the equation

$$\begin{aligned}
&u_{tt} - \Delta u + V(x)\, u = h(x)\, u^3, \quad x \in \Omega, t > 0 \\
&u(x,0) = f(x),\ u_t(x,0) = g(x) \quad x \in \Omega \\
&|x|^2 u + \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega.
\end{aligned}$$

   (a) Show that if $V(x) \geq -\alpha$ for some $\alpha > 0$ and any $x \in \Omega$ and there is a solution $u \in C^2(\overline{\Omega} \times [0,\infty))$, then it is unique.

   (b) In the event $f = 0$ and $h \leq 0$, if $u \in C^2(\overline{\Omega} \times [0,\infty))$ is a solution, show that for all $t > 0$

$$\int_\Omega \left(u_t^2 + |\nabla u|^2 + V(x)\, u^2\right) dx \leq \int_\Omega g^2 dx.$$

7. Consider the equation

$$\begin{aligned}
u_{tt} - \Delta u &= -u, \quad (x,y,t) \in \mathbb{R}^2 \times (0,\infty) \\
u(x,y,0) &= 0, \quad (x,y) \in \mathbb{R}^2 \\
u_t(x,y,0) &= h(x,y), \quad (x,y) \in \mathbb{R}^2
\end{aligned}$$

where h is a smooth function defined on $\mathbb{R}^2$. Find a formula for the solution $u(x,y,t)$.

Hint: Introduce $v(x,y,z,t) = \cos(z)u(x,y,t)$ *defined on* $\mathbb{R}^3 \times (0,\infty)$ *and notice that* $u(x,y,t) = v(x,y,0,t)$.

PDE Preliminary Exam, January 2021

1. Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$. Consider the first-order p.d.e.

$$u_x^2 + u_y^2 = u^2 \quad \text{on } \Omega$$

satisfying $u = 1$ on $x^2 + y^2 = 1$. Prove that there exist exactly two solutions $u \in C^1(\Omega)$. Also find $\lim\limits_{r\to 0} u(x,y),\ r = (x^2+y^2)^{1/2}$.

2. Let $0 < R_1 < R_2,\ \Omega = \{x = (x_1,x_2) \in \mathbb{R}^2 :\ R_1 < |x| < R_2\},\ |x|^2 = x_1^2 + x_2^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u \geq 0$ on Ω. Denote $M(r) = \sup\limits_{|x|=r} u$ for $R_1 \leq r \leq R_2$. Prove

$$M(r) \leq [M(R_1)\ln(R_2/r) + M(R_2)\ln(r/R_1)]\ (\ln(R_2/R_1))^{-1}$$

for $r \in [R_1, R_2]$.
Hint: Consider an auxiliary harmonic function $v(r)$.

3. Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. Assume $b_1, ..., b_n \in C^1(\overline{\Omega})$ and let $Lu = \Delta u + \sum_{i=1}^n b_i(x) u_{x_i}$. Suppose $u \in C^3(\overline{\Omega})$ satisfies $Lu = 0$ on Ω. Define $v = u^2,\ w = |Du|^2 = \sum_{k=1}^n u_{x_k}^2$ on $\overline{\Omega}$.
Prove
(a) $Lv = 2|Du|^2$ on Ω.
(b) For some $M > 0,\ Lw \geq 2|H|^2 - M|Du|^2$ on Ω; here the Hessian $H = [u_{x_k x_i}],\ |H|^2 = \sum_{i,k=1}^n u_{x_k x_i}^2$.
(c) For some $\lambda > 0,\ L(\lambda v + w) \geq 0$ on Ω, and for some $C > 0$

$$||Du||_{L^\infty(\Omega)} \leq\ C(||Du||_{L^\infty(\partial\Omega)} + ||u||_{L^\infty(\partial\Omega)}).$$

4. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u_0 \in C^0(\overline{\Omega}),\ g \in C^0(\mathbb{R}),\ a(x,t) \in C^1(\overline{\Omega} \times [0,T]),\ a \geq 0$ on $\overline{\Omega} \times [0,T]$. Assume $u \in C^2(\overline{\Omega} \times [0,T])$ solves

$$u_t = \mathrm{div}(\mathrm{a}(\mathrm{x},\mathrm{t})\nabla \mathrm{u}) + \mathrm{g}(\mathrm{u})|\nabla \mathrm{u}| \quad \text{on} \quad \Omega \times [0,\mathrm{T}]$$

with initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$, and boundary condition $u(x,t) = 0$ for $(x,t) \in \partial\Omega \times [0,T]$. Prove that $|u(x,t)| \leq \max\limits_{\overline{\Omega}} |u_0|$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.

5. Let u be the bounded solution to the initial value problem

$$u_t = \Delta u \quad \text{on} \quad \mathbb{R}^{\mathrm{n}} \times [0,\infty)$$

with initial condition $u(\cdot, 0) = u_0$ where u_0 is bounded on $\mathbb{R}^n$ and satisfies, for some $\alpha \in (0, 1)$ and $\mathrm{C} > 0$, $|\mathrm{u_0(x)} - \mathrm{u_0(y)}| \leq \mathrm{C}|\mathrm{x} - \mathrm{y}|^\alpha$, $\mathrm{x}, \mathrm{y} \in \mathbb{R}^\mathrm{n}$. Prove that there exists a constant $C_1 > 0$ such that $|u(x,t) - u(x,s)| \leq C_1|t^{\alpha/2} - s^{\alpha/2}|$ for all $x \in \mathbb{R}^n$, $s, t \geq 0$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for every $R > 0$ there exists $N = N(R) > 0$ such that

$$|f(s,t)| \leq N(|s| + |t|) \quad \text{ for all } (\mathrm{s}, \mathrm{t}) \in \mathbb{R}^2,\ |\mathrm{s}| + |\mathrm{t}| \leq \mathrm{R}.$$

Let u be a smooth compactly supported solution of the nonlinear wave equation

$$u_{tt} - \Delta u + f(u, u_t) = 0 \quad \text{on } \ \mathbb{R}^3 \times (0, \infty).$$

Assune that there is $x_0 \in \mathbb{R}^3$ and $t_0 > 0$ such that

$$u(x, 0) = u_t(x, 0) = 0 \quad \text{for all } \mathrm{x} \in \mathrm{B}(\mathrm{x_0}, \mathrm{t_0})$$

($\mathrm{B}(x_0, t_0)$ is the open ball in $\mathbb{R}^3$ with radius t_0 and centered at x_0). Prove that $u = 0$ in the cone $K(x_0, t_0)$ defined by

$$K(x_0, t_0) = \{(x, t) \in \mathbb{R}^4 : 0 \leq t \leq t_0,\ |x - x_0| \leq t_0 - t\}.$$

Hint: One may consider the energy function $e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + |\nabla u|^2 + u^2)\, dx$.

7. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 1$ if $|x| < 1$, $g(x) = 0$ if $|x| \geq 1$. Use d'Alembert's formula to find the solution u of the wave equation

$$u_{tt} - u_{xx} = 0 \quad \text{ on } \mathbb{R} \times (0, \infty)$$

with $u(x, 0) = x^2$ and $u_t(x, 0) = g(x)$, $x \in \mathbb{R}$. Show that u is not differentiable with respect to the variable t at $(x_0, t_0) = (0, 1)$.

PDE Preliminary Exam, August 2020

1. Let $\Omega = \{(x,t) : x > 0,\ t > 0\}$. Assume $f \in C^\infty(\overline{\Omega})$, f has bounded support and $f = 0$ on $\{t = 0\}$. Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$u_t + u_x + u = f(x,t) \text{ on } \Omega,$$

$$u = 0 \text{ on } \{\mathrm{x} = 0\} \cup \{\mathrm{t} = 0\}.$$

(a) For each $t > 0$, prove that $u(\cdot, t)$ has bounded support.
(b) For each $t > 0$, prove

$$\int_0^\infty u_t^2 \, dx \le \int_0^t e^{s-t} \int_0^\infty f_t^2(x,s) \, dx \, ds.$$

(c) Prove there exists $K > 0$ such that $\int_0^\infty u_t^2 \, dx \le K e^{-t}$ for all $t > 0$.

2. Let $a > 0$, $\Omega = (-1,1) \times (-a,a) \subset \mathbb{R}^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\Delta u = -1 \text{ on } \Omega, \quad \mathrm{u} = 0 \text{ on } \partial\Omega.$$

Using the functions $v(x,y) = (1 - x^2)(a^2 - y^2)$, $w(x,y) = 2 - x^2 - \frac{y^2}{a^2}$ (or constant multiples of them), find positive bounds $C_1(a)$ and $C_2(a)$ such that

$$C_1(a) \le u(0,0) \le C_2(a).$$

3. Suppose $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ is open, bounded with C^∞-smooth boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$-\Delta(u^3) = u \text{ on } \Omega, \quad \mathrm{u} = 0 \text{ on } \partial\Omega.$$

(a) Using the Green's function show there exists a constant $C > 0$ depending only on Ω, but not on the solution, such that $\int_\Omega |u(x)|^3 \, dx \le C$, and $\sup_\Omega |u| \le C$.
(b) Show that, if $u \ge 0$ on Ω, then either, $u \equiv 0$ on Ω or $\mathrm{u} > 0$ on Ω.
(c) Let v be the eigenfunction corresponding to the first (least) eigenvalue λ of $-\Delta v = \lambda v$ on Ω, $\mathrm{v} = 0$ on $\partial\Omega$ (recall $v > 0$ on Ω). Show that, if $u \ge v$, then $u^3 \ge \frac{1}{\lambda} v$.

(d) Assuming also $u^3 \in C^1(\overline{\Omega})$, prove $\int_\Omega |\nabla(u^2)|^2 \ dx = C_1 \int_\Omega u^2 \ dx \leq C_2$ where $C_1,\ C_2$ depend only on Ω, not on u.

4. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be smooth and compactly supported, and

$$m = \int_{\mathbb{R}^n} u_0(y) \ dy.$$

Let u be a solution of the Cauchy problem

$$u_t - \Delta u = 0 \ \text{ on } \ \mathbb{R}^{\rm n} \times (0, \infty),$$

$$u(x, 0) = u_0(x) \ \ x \in \mathbb{R}^n,$$

with $|u(x,t)| \leq Ae^{a|x|^2}$ for some fixed $A, a > 0$ and all $(x.t) \in \mathbb{R}^n \times (0, \infty)$. Prove that there is a constant N depending only on n such that

$$\sup_{x \in \mathbb{R}^n} |u(x,t) - m \ \Phi(x,t)| \leq \frac{N}{t^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} |y||u_0(y)| \ dy, \quad \text{for all} \ \ \mathrm{t} > 0,$$

where $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$.

5. Let u be a smooth function on $\overline{B}_1 \times [0,1]$ that satisfies the equation

$$a_0 \ u_t - b_0 \ \Delta u + u = 1 \ \text{ on } \ \mathrm{B}_1 \times (0,1),$$

$$u = 1 \ \text{ on } \ \partial \mathrm{B}_1 \times (0,1),$$

$$u(x, 0) = 1 \ \ x \in B_1,$$

where $a_0, b_0 : \overline{B}_1 \times [0,1] \to [0, \infty)$ are given continuous functions (B_1 = unit ball in $\mathbb{R}^n$). Prove that $u \leq 1$ on $\overline{B}_1 \times [0,1]$.

6. Assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞-smooth boundary $\partial\Omega$. Let $\mathrm{T} > 0$, $\ \Omega_T = \Omega \times (0, T]$. Suppose $a \in C^1(\overline{\Omega})$, $a > 0$ on $\overline{\Omega}$, $\ \phi, \psi \in C^2(\overline{\Omega})$. Suppose $u \in C^2(\overline{\Omega_T})$ is a solution of

$$u_{tt} - a(x) \Delta u = u^3 \ \text{ on } \ \Omega_{\mathrm{T}},$$

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega \times [0, \mathrm{T}],$$

$$u = \phi, \ \ u_t = \psi \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\}.$$

Prove that u is unique.

7. Assume $\phi \in C^2(\mathbb{R})$ and $h, \psi \in C^1(\mathbb{R})$. Consider the initial-value problem with $u \in C^2(\mathbb{R} \times [0, \infty))$

$$u_{tt} - u_{xx} = h(x - t) \ \text{ on } \ \mathbb{R} \times [0, \infty), \tag{1}$$

$$u = \phi(x), \quad u_t = \psi(x) \ \text{ at } \ \mathrm{t} = 0, \quad \mathrm{x} \in \mathbb{R}. \tag{2}$$

(a) Find a solution of the p.d.e. in (1).
(b) Find a solution of (1) and (2).

UTK PDE Prelim Exam, Spring 2020

Question 1: Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function. Find solutions of the following initial-value problem in $\mathbb{R}^2$

$$u_x + (1+x^2)u_y - u = 0 \quad \text{with} \quad u(x, \frac{1}{3}x^3) = g(x).$$

Question 2: Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the following equation in $\mathbb{R}^2$

$$xu_x + yu_y = 2u \quad \text{with} \quad u(x,0) = h(x).$$

(a) Check that the line $\{y = 0\}$ is characteristic at each point and find all h satisfying the compatibility condition on $\{y = 0\}$.

(b) For h as compatible in (a), solve the PDE.

Question 3: Let ϕ be smooth, compactly supported function defined in the unit ball $B_1 \subset \mathbb{R}^n$ such that $\phi = 1$ on $B_{1/2}$, where $B_{1/2} \subset \mathbb{R}^n$ is the ball of radius $1/2$ centered at the origin. Suppose that u is harmonic in B_1.

(a) Prove that there is $\alpha > 0$ depending only on n and $\sup |\Delta \phi|$ and $\sup |\nabla \phi|$ such that

$$\Delta(\phi^2 |\nabla u|^2 + \alpha u^2) \geq 0 \quad \text{in} \quad B_1.$$

(b) Use part (a) and the maximum principle to conclude that there is a constant $C > 0$ depending only on n, ϕ such that

$$\sup_{B_{1/2}} |\nabla u| \leq C \sup_{\partial B_1} |u|.$$

Question 4: Let $B_1 \subset \mathbb{R}^2$ be the unit ball with boundary ∂B_1. Let $f, c \in C(\overline{B}_1)$ and $g \in C(\partial B_1)$. Assume that $c(x,y) > 0$ for all $(x,y) \in B_1$. Prove that there exists at most one C^2-solution to the following equation

$$\begin{cases} -x^2 u_{xx} - y^2 u_{yy} + c(x,y)u &= f \quad \text{in} \quad B_1 \\ u &= g \quad \text{on} \quad \partial B_1. \end{cases}$$

Question 5: Let a_0 be a smooth and compactly supported function defined on $\mathbb{R}^n$ and $p_0 \in (1, \infty)$. Consider the following Cauchy problem

$$\begin{cases} u_t - \Delta u &= |u|^{p_0 - 1} u \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u(x,0) &= a_0(x) \quad x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Define the scaling

$$u_\lambda(x,t) = \lambda^\beta u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

(a) Find β (possibly depending on n, p_0) so that if u is a solution of (1), then u_λ is also a solution (1) (with appropriate scaled initial data a_0^λ).

(b) Recall that the L^p-norm is defined by

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

For β found in a), find p so that if u is a solution of (1) then

$$\|u(\cdot, \lambda^2 t)\|_{L^p(\mathbb{R}^n)} = \|u_\lambda(\cdot, t)\|_{L^p(\mathbb{R}^n)}$$

for all $\lambda > 0$ and for all $t > 0$.

Question 6: Let us denote $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$ and $B_1^+ = B_1 \cap \mathbb{R}^2_+$, where B_1 is the unit ball in $\mathbb{R}^2$. Assume that $u = u(x, y, t)$ is a smooth function defined on $\overline{B_1}^+ \times [0, 1]$ and satisfying

$$u_t - y^\alpha [u_{xx} + u_{yy}] + u_y + u \leq 0 \quad \text{for} \quad (x, y) \in B_1^+ \quad \text{and} \quad t \in (0, 1),$$

where $\alpha > 0$ is a given number. Assume that $u(x, y, 0) \leq 0$, and that $u \leq 0$ on $(\partial B_1 \cap \mathbb{R}^2_+) \times (0, 1)$, where ∂B_1 denotes the boundary of B_1. Prove that

$$u \leq 0 \quad \text{on} \quad \overline{B}_1^+ \times [0, 1].$$

Note: We are not given any information on the boundary data on the part of the boundary where $y = 0$.

Question 7: Let $u_1(x)$ and $u_2(x)$ be smooth functions whose supports are in the unit ball $B_1 \subset \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$ and each $t_0 > 0$, let $C(x_0, t_0)$ be the cone defined by

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, \quad |x - x_0| \leq t_0 - t\}.$$

Assume that $u \in C^2$ is the solution of the equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty)$$

with given initial data $u(x, 0) = u_1(x)$ and $u_t(x, 0) = u_2(x)$.

Give the proof for the finite propagation speed result for the wave equation, namely $u = 0$ on $C(x_0, t_0)$ for all $x_0 \in \mathbb{R}^n$ with $|x_0| > 1$ and $t_0 = |x_0| - 1$.

Question 8: Let u be a smooth solution of the equation

$$u_{tt} - \Delta u = f \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $u(\cdot, 0) = u_t(\cdot, 0) = 0$. Also, let v be a smooth solution of the equation

$$v_{tt} - \Delta v = g \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $v(\cdot, 0) = v_t(\cdot, 0) = 0$. Assume that $|f|^2 \leq g$. Prove that $2u(x, t)^2 \leq t^2 v(x, t)$ for all $x \in \mathbb{R}^3$ and $t > 0$.

PDE Prelim Exam, Fall 2019

Question 1: Solve the Cauchy problem

$$\begin{cases} xu_x - yu_y = u - y, & x > 0, y > 0, \\ u(y^2, y) = y, & y > 0. \end{cases}$$

Question 2: Let a, R be positive numbers and consider the equation

$$\begin{cases} u_t + au_x &= f(x,t), & 0 < x < R, \quad t > 0, \\ u(0,t) &= 0, & t > 0, \\ u(x,0) &= 0, & 0 < x < R. \end{cases}$$

Prove that for each solution $u(x,t) \in C^1((0,R) \times (0,\infty))$ we have

$$\int_0^R u^2(x,t)dx \le e^t \int_0^t \int_0^R f^2(x,s)dxds, \quad \forall\, t > 0.$$

Question 3: Let $r > 0$ and let f, g be continuous functions defined on $\overline{B}_r(0)$. Let u be in $C^2(B_r(0)) \cap C(\overline{B}_r(0))$ be the solution of the equation

$$\begin{cases} -\Delta u &= f, & B_r(0), \\ u &= g, & \partial B_r(0). \end{cases}$$

Prove that

$$u(0) = \fint_{\partial B_r(0)} g(x)dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left[\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right] f(x)dx.$$

Hint: Consider

$$\phi(s) = \fint_{\partial B_s(0)} u(y)dS, \quad 0 < s \le r.$$

Compute $\phi'(s)$ and then find $\phi(0)$.

Question 4: Let $R > 0$ and we denote B_R the ball of radius R centered at the origin in $\mathbb{R}^n$. Let c, f be continuous functions on $\overline{B}_R$. Assume that $c \le 0$ on $\overline{B}_R$, and also assume that $u \in C^2(B_R) \cap C(\overline{B}_R)$ satsifies

$$\begin{cases} \Delta u + cu &= f & \text{in} \quad B_R, \\ u &= 0 & \text{on} \quad \partial B_R. \end{cases}$$

Prove that

$$\sup_{B_R} |u| \le \frac{R^2}{2n} \sup_{B_R} |f|$$

Hint: Let $A = \sup_{B_R} |f|$ and

$$v(x) = \frac{AR^2}{2n}(R^2 - |x|^2)$$

Use the maximum principle to prove that $|u(x)| \le v(x)$ on B_R.

Question 5: Let u_0 be the smooth and compactly supported function defined on $\mathbb{R}^n$. Assume that u is a solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u &= 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Let $p, q \in (1, \infty)$ with $p \geq q$ and consider the inequality

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq \frac{N}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^n)}, \quad t > 0$$

with $N = N(n, p, q)$ and $\alpha = \alpha(n, p, q)$, where we denote

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x, t)|^p dx \right)^{\frac{1}{p}}$$

and similar notation is also used for $\|u_0\|_{L^q(\mathbb{R}^n)}$.

Use the scaling property of the heat equation to find the number α (certainly, show all of the work).

Question 6: Assume that u is a smooth, bounded solution of the equation

$$\begin{cases} u_t - \Delta u &= u(1-u) & \text{in} \quad B_1 \times (0, 1] \\ u &= 0 & \text{on} \quad \partial B_1 \times (0, 1] \\ u &= \frac{1}{2} & \text{on} \quad B_1 \times \{0\}. \end{cases}$$

Prove that $0 \leq u \leq 1$.

Question 7: Let φ be a smooth, compactly supported function on $\mathbb{R}^2$. Assume that u is a smooth solution of

$$\begin{cases} u_{tt} - \Delta u &= 0 & \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\ u(\cdot, 0) &= 0 & \text{on} \quad \mathbb{R}^2, \\ u_t(\cdot, 0) &= \varphi & \text{on} \quad \mathbb{R}^2. \end{cases}$$

Prove that

$$|u(x, t)| \leq \frac{1}{2\sqrt{t}} \Big(\|\varphi\|_{L^1(\mathbb{R}^2)} + \|\nabla \varphi\|_{L^1(\mathbb{R}^2)} \Big), \quad \forall\, t > 1.$$

Question 8: Assume that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of the wave equation

$$u_{tt} = \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0, \infty).$$

Let

$$E(t) = \frac{1}{2} \int_{B_{1-t}} \Big[|u_t(x, t)|^2 + |\nabla u(x, t)|^2 \Big] dx \quad \text{for} \quad t \in (0, 1),$$

where $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$ and B_r denotes the ball in $\mathbb{R}^n$ with radius $r > 0$ and centered at the origin.

(a) Prove that

$$E'(t) = \int_{B_{1-t}} \Big[u_t(x, t) u_{tt}(x, t) + \sum_{i=1}^n u_{x_i} u_{x_i t} \Big] dx$$
$$- \frac{1}{2} \int_{\partial B_{1-t}} \Big[u_t^2(x, t) + |\nabla u(x, t)|^2 \Big] dS(x).$$

(b) Use the note that

$$\Big[u_{x_i} u_t \Big]_{x_i} = u_{x_i} u_{x_i t} + u_{x_i x_i} u_{tt}.$$

to prove that $E'(t) \leq 0$. Then, conclude also that $u = 0$ on $\{(x, t) : |x| \leq 1 - t,\ 0 \leq t \leq 1\}$ if $u(x, 0) = u_t(x, 0) = 0$ for $x \in B_1$.

PDE Preliminary Exam, August 2018 — UTK

Question 1: For $x > 0$, consider the equation:

$$\begin{cases} uu_x + 2xu_y = 0 \text{ in } \mathbb{R}^2 \\ u(x,0) = \frac{1}{x} \text{ for } x > 0. \end{cases}$$

For $t_0, t_1 > 0$ with $t_0 \neq t_1$, let C_0 be the characteristic passing through the point $(t_0, 0, 1/t_0)$ and let C_1 be the characteristic passing through $(t_1, 0, 1/t_1)$. Determine whether the projections of C_0 and C_1 onto the x-y plane intersect for some $y > 0$ (i.e., whether a shock develops), and if they do, find the point (x, y) of intersection.

Question 2: Given a bounded domain Ω in $\mathbb{R}^n$, let h be the solution to

$$\Delta h = -1 \quad \text{in} \quad \Omega\,, \qquad h = 0 \quad \text{on} \quad \partial\Omega\,.$$

Let $a > 0$ be a constant.

Prove: If there exists a function $u > 0$ that satisfies the equation

$$\Delta u = \frac{1}{u} \quad \text{in} \quad \Omega\,, \qquad u \equiv a \quad \text{on} \quad \partial\Omega\,,$$

then $a \geq \sqrt{\max_{\overline{\Omega}} h}$.

Hint: Prove $u \leq a$. Then prove a better upper bound for u.

Question 3:

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and even (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$). Suppose $u = u(x,t) \in C_1^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ satisfies

$$\begin{cases} u_t = u_{xx} & \text{for } x \in \mathbb{R}, 0 < t < \infty, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\ |u(x,t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}, 0 < t < \infty, \end{cases}$$

for some positive constants K and a. Prove that for each $t > 0$, $u(x,t)$ is an even function of x: i.e., $u(-x,t) = u(x,t)$ for all $t > 0$.

(b) Assume $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded. For $x \geq 0$ and $t \geq 0$, suppose $u = u(x,t) \in C^2([0,\infty) \times [0,\infty))$ satisfies

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < \infty, 0 < t < \infty, \\ u(x,0) = f(x) & \text{for } 0 \leq x < \infty, \\ u_x(0,t) = 0 & \text{for } 0 < t < \infty \\ |u(x,t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}_+, 0 < t < \infty, \end{cases}$$

for some positive constants K and a. Here $u_x(0,t)$ is interpreted as the x-derivative of u from the right at $(0,t)$. Find a function $H = H(x,y,t)$ such that

$$u(x,t) = \int_0^\infty H(x,y,t) f(y)\, dy,$$

and justify your answer.

Pg. 2 - DE
Aug. 2018

Question 4: Consider the nonlinear PDE

$$u_{tt} - \Delta u + u^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}.$$

1. Assume that u is smooth and has compact support in x for each t. What is the energy expression

$$E(t) = \int_{\mathbb{R}^3} q(u, u_t, \nabla u) dx$$

which is conserved, i.e., $E'(t) = 0$?

2. For any $\alpha > 0$, and $x_0 \in \mathbb{R}^3$, denote by

$$E_\alpha(t) = \int_{B_\alpha(x_0)} q(u, u_t, \nabla u) dx$$

the energy contained in the ball of radius $\alpha > 0$ centered at x_0. Show that for any $T > 0$ and $a > 0$,

$$E_a(T) \leq E_{a+T}(0)$$

Hint: Work with the 'energy'

$$\tilde{E}(t) := \int_{B_{T+a-t}(x_0)} q(u, u_t, \nabla u) dx$$

3. Given $a > 0$, show that if $u(x, 0) = u_t(x, 0) = 0$ for $|x| > a$, then $u(x, t) = 0$ for all $|x| \geq a + t$, $t \geq 0$.

Question 5: Let B be the unit ball in $\mathbb{R}^n$ and let $u \in C^\infty(\bar{B} \times [0, \infty))$ satisfy

$$\begin{aligned} &u_t - \Delta u + u^{1/2} = 0 && \text{on } B \times (0, \infty) \\ &0 \leq u && \text{on } B \times (0, \infty) \\ &u = 0 && \text{on } \partial B \times (0, \infty). \end{aligned}$$

(a) Show that, if $u|_{t=t_0} \equiv 0$, then $u \equiv 0$ for $t > t_0$ as well.

(b) Prove that there is a number T depending only on $M := \max u|_{t=0}$ such that $u \equiv 0$ on $B \times (T, \infty)$.

Hint: Let v be the solution of the IVP,

$$\frac{dv}{dt} + v^{\frac{1}{2}} = 0, \quad v(0) = M,$$

and consider the function $w = v - u$.

Question 6:

(a) Find a C^1 solution in $\mathbb{R}^+ \times \mathbb{R} \ni (x, y)$ to:

$$x^2 u_x - y^2 u_y = u^2 \text{ for } x > 0, y \in \mathbb{R}, \qquad u(1, y) = \frac{1}{1 + y^2}$$

(b) Explain why this solution is not unique as a solution in $C^1(\mathbb{R}^+ \times \mathbb{R})$, but its restriction to some appropriate open set U containing the initial curve $\{1\} \times \mathbb{R}$ is unique in $C^1(U)$.

Question 7: Suppose $f, g \in C^\infty(\mathbb{R}^n)$. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfies

$$\begin{aligned} u_{tt} &= \Delta u, && (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) &= f(x), && x \in \mathbb{R}^n, \\ u_t(x,0) &= g(x), && x \in \mathbb{R}^n. \end{aligned}$$

Prove that

$$\int_{\mathbb{R}^n} u(x,t)\,dx = C_1 t + C_2,$$

for all $t > 0$, where $C_1 = \int_{\mathbb{R}^n} g(x)\,dx$ and $C_2 = \int_{\mathbb{R}^n} f(x)\,dx$, under either of the two conditions:

(i) $n = 3$, $\int_{\mathbb{R}^3} |f(x)|\,dx < \infty$, $\int_{\mathbb{R}^3} |\nabla f(x)|\,dx < \infty$, and $\int_{\mathbb{R}^3} |g(x)|\,dx < \infty$; or

(ii) $n \in \mathbb{N}$, and f and g have compact support.

Question 8: Let $u \in C^2(\mathbb{R}^n)$ be a subharmonic function and consider the spherical averages

$$v(r) := \fint_{\partial B_r(0)} u(x)\,dS(x)\,.$$

(a) Show that the function $x \mapsto v(|x|)$ is also subharmonic in $\mathbb{R}^n$, and that $r \mapsto r^{n-1}v'(r)$ is monotonic.

(b) Now let $n = 2$. Prove that, if u is also bounded, then u is a constant.

PDE Preliminary Exam, January 2018

Instruction:

Solve all eight problems. Begin your answer to each question on a separate sheet. Explain all your steps.

1. A smooth function u defined in the first quadrant on the xy–plane satisfies

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = -2u, \qquad u(x,0) = x.$$

   Determine $u(0,y)$.

2. Suppose that $u(x,t)$ is a smooth solution of

$$\begin{cases} u_t + uu_x = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R} \end{cases}$$

   Assume that f is a C^1 function such that

$$f(x) = \begin{cases} 0 & \text{for } x < -1 \\ 1 & \text{for } x > 1 \end{cases} \quad \text{and } f'(x) > 0, \text{ for } |x| < 1.$$

   (a) Sketch the characteristics emanating from $(x_0, 0)$ for several values of $x_0 < -1, x_0 \in (-1,1)$, and $x_0 > 1$.

   (b) Show that for $t > 0$,

$$\lim_{r\to\infty} u(rx, rt) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < t \\ 1 & \text{for } x > t \end{cases}$$

3. Suppose that for all $r > 2$, there exists a function $u_r : \mathbb{R}^3 \to \mathbb{R}$ that is continuous and satisfies

$$\begin{cases} \Delta u = 0 & \text{in } B_r(0) \setminus \overline{B_1(0)} \\ u(x) = 0 & \text{for } |x| \geq r \\ u(x) = 1, & \text{for } x \in \overline{B_1(0)}. \end{cases}$$

   (a) Show that for all $x \in \mathbb{R}^3$, if $2 < r_1 \leq r_2$, then

$$0 \leq u_{r_1}(x) \leq u_{r_2}(x) \leq 1.$$

(b) Show that

i. $u(x) = \lim_{r\to\infty} u_r(x)$ is harmonic on $\mathbb{R}^3 \setminus \overline{B_1(0)}$

ii. $\lim_{|x|\to\infty} u(x) = 0$.

[Hint: noting that $\frac{1}{|x|}$ is harmonic, study $u_r(x) - \frac{1}{|x|}$ over an annulus.]

4. Denote by $\mathbb{R}^n_+ = \{\mathbf{x} = (\mathbf{x}', x_n) : x_n > 0\}$, $\Sigma = \{\mathbf{x} = (\mathbf{x}', x_n) : x_n = 0\}$.
Suppose that u is harmonic in $\mathbb{R}^n_+$, continuous on $\mathbb{R}^n_+ \cup \Sigma$, and $u = 0$ on Σ. Define

$$\overline{u}(x', x_n) := \begin{cases} u(x', x_n) & \text{for } x_n \geq 0, \\ -u(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Then show that $\overline{u}$ is harmonic in $\mathbb{R}^n$.

5. Let $\Omega \subseteq \mathbb{R}^n$ be a C^∞ bounded domain. Assume that $u_0 \in C^\infty(\overline{\Omega})$, $a \in C([0,\infty))$, and $\lim_{t\to\infty} a(t) \leq 0$. Suppose also $u \in C^2(\overline{\Omega} \times [0,\infty))$ satisfies

$$\begin{cases} u_t = \Delta u + a(t)u & \text{on } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = u_0 & \Omega \times \{t = 0\} \end{cases}$$

Prove that

$$\lim_{t\to\infty} \int_\Omega u^2(x,t)dx = 0$$

(Hint: Use the Energy method. You may apply Poincaré's inequality.)

6. Let $\Omega \subseteq \mathbb{R}^n$ be a C^∞ bounded domain, $T > 0$, and $\mathbf{a} \in \mathbb{R}^n$ is a given vector. Suppose $u \in C^2(\overline{\Omega} \times [0,T])$ satisfies

$$\begin{cases} u_t = \Delta u + \mathbf{a} \cdot \nabla u + u^2 & \text{on } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times (0,T], \\ u = 0 & \Omega \times \{t = 0\}. \end{cases}$$

Prove that

(a) $u \geq 0$, on $\Omega \times (0,T]$,

(b) $u_t \geq 0$ on $\Omega \times (0,T]$.
(Hint: What equation does u_t solve?)

7. Let $\Omega \subseteq \mathbb{R}^n$ be a C^∞ bounded domain and let $T > 0$. Suppose $V = V(x)$ and $h = h(x)$ are continuous functions on $\overline{\Omega}$, with $V(x) \geq 0$. Suppose $u = u(x,t) \in C^2(\overline{\Omega} \times [0,T])$, where $x \in \Omega$ and $t \in [0,T]$, and u satisfies

$$\begin{cases} u_{tt} - \Delta u + V(x)u = h(x) & \text{on } \Omega \times (0,T); \\ u(x,0) = 0 & \text{on } \Omega; \\ u_t(x,0) = 0 & \text{on } \Omega; \\ u = -D_{\vec{n}} u & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $D_{\vec{n}}u$ is the outward normal derivative of u on $\partial\Omega$.

(a) Prove that $\int_\Omega h(x)u(x,t)\,dx \geq 0$ for all $t \geq 0$.

Hint: Consider

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 + Vu^2 - 2hu\,dx + \frac{1}{2}\int_{\partial\Omega} u^2\,d\sigma,$$

where $d\sigma$ is surface measure on $\partial\Omega$.

(b) Suppose in addition that $V(x) \geq A$ and $|h(x)| \leq B$, for all $x \in \Omega$, for some constants $A > 0$ and $B > 0$. Prove that

$$\int_\Omega |u(x,t)|\,dx \leq \frac{2B|\Omega|}{A},$$

for all $t \geq 0$, where $|\Omega| = \int_\Omega dx$ is the measure of Ω.

Hint: Start by writing $\int_\Omega |u|\,dx = \int_\Omega \frac{\sqrt{V}|u|}{\sqrt{V}}\,dx$, and apply Cauchy Schwartz.

8. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ is a solution of

$$\begin{cases} u_{tt} = \Delta u & \text{on } \mathbb{R}^n \times (0,\infty); \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n; \\ u_t(x,0) = g(x) & \text{on } \mathbb{R}^n, \end{cases}$$

where $f, g \in C^\infty(\mathbb{R}^n)$ have compact support: there exists $R > 0$ such that $f(x) = 0$ and $g(x) = 0$ if $|x| > R$. Consider the statement:

(S): *For all such f, g and R, and all $x_0 \in \mathbb{R}^n$, there exists $T = T(x_0, R) > 0$ such that $u(x_0, t) = 0$ for all $t > T$.*

(a) Is (S) true if $n = 1$? Either prove (S) or give an example showing that S fails.

(b) Is (S) true if $n = 3$? Either prove (S) or give an example showing that S fails.

PDE Preliminary Exam, August 2017

1. For a given continuous function f, solve the initial-boundary value problem

$$\begin{cases} u_t + (x+1)^2 u_x = x, & \text{for } x > 0, t > 0 \\ u(x,0) = f(x), & x > 0 \\ u(0,t) = -1 + t, & t > 0. \end{cases}$$

Find a condition on f so that the solution $u(x,t)$ is continuous on the first quadrant of $\mathbb{R}^2$, i.e. the region $\{(x,t) \in \mathbb{R}^2 : x > 0, t > 0\}$.

2. Determine an integral (weak) solution to the Burger's equation

$$u_t + (\frac{1}{2}u^2)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

with initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1. \end{cases}$$

3. Let $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary. Suppose p and q are non-negative continuous functions defined on Ω, satisfying $p(x) + q(x) > 0$ (strict inequality) for all $x \in \Omega$. Find all functions $u \in C^2(\overline{\Omega})$ satisfying

$$\begin{cases} \triangle u = pu^3 + qu & \text{on } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{n}(x)$ is the outward unit normal to Ω at $x \in \partial\Omega$.

4. Suppose u is harmonic on a C^∞ domain $\Omega \subseteq \mathbb{R}^n$, and let $u(x) = 0$ for $x \notin \Omega$. Suppose φ is a C^∞ function on $\mathbb{R}^n$ such that $\varphi(x) = 0$ if $|x| \geq 1$, and φ is radial: there exists a function $\varphi_0 : [0,\infty) \to \mathbb{R}$ such that $\varphi(x) = \varphi_0(|x|)$. For $\epsilon > 0$, let

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi\left(\frac{x}{\epsilon}\right).$$

Let

$$A = \int_{\mathbb{R}^n} \varphi(x)\, dx.$$

Fix $x_0 \in \Omega$ and let $R > 0$ be such that $x \in \Omega$ if $|x - x_0| < R$. For $0 < \epsilon < R$, prove that

$$\varphi_\epsilon * u(x_0) = Au(x_0),$$

where $*$ denotes convolution: by definition, $f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$.

5. Suppose that $\mathbf{b} \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$ are given. Consider the Cauchy problem

$$(*) \qquad \begin{cases} u_t + \mathbf{b} \cdot \nabla u + \beta u = \Delta u, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

(a) Determine $\mathbf{a} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that if u is a smooth solution to $(*)$, then $v(x, t) = e^{-(\mathbf{a} \cdot x + \alpha t)} u(x, t)$ solves the Cauchy problem

$$\begin{cases} v_t = \Delta v, & \text{in } \mathbb{R}^n \times (0, \infty) \\ v(x, 0) = e^{-\frac{\mathbf{b}}{2} \cdot x} f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

(b) Write down an explicit formula for a solution $u(x, t)$ to (*).

6. Let $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, and $T > 0$. Denote the cylinder $\Omega_T = \Omega \times (0, T]$ and its parabolic boundary $\partial_p \Omega_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{0\})$.

(a) Prove the following version of the maximum principle. Suppose that u and v are two functions in $C^2(\overline{\Omega_T})$ such that

$$\begin{aligned} u_t - \Delta u \leq v_t - \Delta v \quad & \text{in } \Omega_T \\ u \leq v \quad & \text{on } \partial_p \Omega_T. \end{aligned}$$

Then $u \leq v$ in Ω_T.

(b) Suppose that $f(x, t)$, $u_0(x)$ and $\phi(x, t)$ are continuous functions in their respective domains. Let $u \in C^2(\overline{\Omega_T})$ satisfy

$$\begin{cases} u_t - \Delta u = e^{-u} - f(x, t), & \text{in } \Omega_T \\ u|_{t=0} = u_0, & \text{in } \Omega \\ u|_{\partial\Omega \times (0,T)} = \phi. \end{cases}$$

Let $a = \|f\|_{L^\infty}$ and $b = \sup\{\|u_0\|_{L^\infty}, \|\phi\|_{L^\infty}\}$.

i. Show that $-(aT + b) \leq u(x, t)$, for all $(x, t) \in \overline{\Omega_T}$.
Hint: Introduce $v(x, t) = -(at + b)$ and use part a).

ii. Prove $u(x, t) \leq T\, e^{aT+b} + aT + b$, for all $(x, t) \in \overline{\Omega_T}$

7. Suppose that $f \in C^2(\mathbb{R})$ is odd and $2-$periodic (i.e. $f(x+2) = f(x)$ for all $x \in \mathbb{R}$). Let $u \in C^2([0,1] \times \mathbb{R})$ solve

$$\begin{cases} u_{tt} - u_{xx} = \sin(\pi x) \quad \text{in } (0,1) \times \mathbb{R} \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \quad x \in [0,1] \\ u(0,t) = 0 = u(1,t), \quad t \in \mathbb{R}. \end{cases}$$

(a) Prove uniqueness of the solution $u \in C^2([0,1] \times \mathbb{R})$.

(b) Find the solution u, and show that it satisfies $u(x,t+2) = u(x,t)$, and $u(x,-t) = u(x,t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$.

8. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded with C^∞-smooth boundary $\partial\Omega$. Let $T > 0$, and denote $\Omega_T = \Omega \times (0,T]$. Suppose also that $f \in C^1(\mathbb{R}^{n+2})$, $\phi, \psi \in C^2(\overline{\Omega})$, and $u \in C^2(\overline{\Omega_T})$ is a solution of

$$\begin{cases} u_{tt} - \Delta u = f(u, u_t, \nabla u), \qquad \text{in } \Omega_T \\ u = \phi, \quad u_t = \psi, \qquad \text{on } \Omega \times \{t = 0\}, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, \qquad \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Prove that u is unique.

Hint: You may use an energy function of the form

$$E(t) = \frac{1}{2} \int_\Omega (w_t^2 + |\nabla w|^2 + w^2) dx.$$

PDE Qualifying Exam Fall 2016

1.) Consider the PDE, for $x \in \mathbb{R}$ and $y \in \mathbb{R}$:

$$(*) \qquad \begin{cases} 2yu_x + u_y = u^4, \\ u(x,0) = f(x), \end{cases}$$

for some C^2 function f.

(a) Show that $(*)$ has a solution that exists for all $x \in \mathbb{R}$ and all $y > 0$ if and only if $f(t) \leq 0$ for all $t \in \mathbb{R}$.

(b) Show that if $(*)$ has a solution for all $(x,y) \in \mathbb{R}^2$, then $f(t) = 0$ for all t and u is identically 0.

2.) Suppose $n \geq 2$, $R > 0$, $B(0,R) \subseteq \mathbb{R}^n$, and $u : \overline{B(0,R)} \to \mathbb{R}$ satisfies $u \in C(\overline{B(0,R)})$, u is harmonic on $B(0,R)$, and $u \geq 0$ on $B(0,R)$.

(a) Prove that

$$\frac{(R-|x|)R^{n-2}}{(R+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{(R+|x|)R^{n-2}}{(R-|x|)^{n-1}}u(0),$$

for all $x \in B(0,R)$.

(b) Prove that

$$|u_{x_j}(x)| \leq \frac{(2n+2)R^{n-1}}{(R-|x|)^n}u(0),$$

for $x \in B(0,R)$ and $j = 1, 2, \ldots, n$.

3.) Suppose $n \geq 3$, and $\Omega \subseteq \mathbb{R}^n$ is a C^∞ bounded domain. Let

$$\Gamma(x) = \frac{1}{(2-n)\omega_n |x|^{n-2}},$$

for $x \in \mathbb{R}^n \setminus \{0\}$, be the fundamental solution for the Laplacian on $\mathbb{R}^n$. Let $G(x,y)$ be the Green's function for the Laplacian on Ω (i.e., $G(x,y) = h(x,y) + \Gamma(x-y)$, where, for each $x \in \Omega$, $h(x,y)$ is a harmonic function of y on Ω, and $h(x,y) = -\Gamma(x-y)$ for $x \in \Omega$ and $y \in \partial\Omega$). You can assume that $G \in C^2(\overline{\Omega} \times \overline{\Omega} \setminus \{(x,y) \in \overline{\Omega} \times \overline{\Omega} : x = y\})$. Prove that $\Gamma(x-y) < G(x,y) < 0$, for $(x,y) \in \Omega \times \Omega$ with $x \neq y$.

4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^1 domain and suppose $T > 0$. Let $\Omega_T = \Omega \times (0, T]$. Suppose $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 - u(u-1)(u-2), & \text{for } (x,t) \in \Omega_T, \\ u(x,t) = e^{-t}[1 + \sin(|x|^2)], & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = 1 + \sin(|x|^2), & \text{for } x \in \Omega. \end{cases}$$

Prove that $0 \le u \le 2$ on $\overline{\Omega_T}$.

5.) Suppose $g = g(x,t) \in C_1^2(\overline{\mathbb{R}_+^{n+1}})$, where $x \in \mathbb{R}^n$ and $t \ge 0$, and suppose g has compact support. Suppose $u \in C_1^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$ satisfies, for some positive constants K and a,

$$\begin{cases} u_t - \Delta u = g(x,t) & \text{for } x \in \mathbb{R}^n, t \in (0,\infty), \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}^n, \\ |u(x,t)| \le Ke^{a|x|^2} & \text{for } x \in \mathbb{R}^n, t \in [0,\infty). \end{cases}$$

Suppose $p > n/2$ and $M = \max_{t \ge 0} \int_{\mathbb{R}^n} |g(x,t)|^p \, dx$. Prove that there exists a constant C, depending only on n and p, such that

$$|u(x,t)| \le CM^{1/p} t^{1 - \frac{n}{2p}},$$

for all $(x,t) \in \mathbb{R}_+^{n+1}$.

6.) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, and $g : \mathbb{R}^3 \to \mathbb{R}$ is C^∞. Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ satisfies

$$\begin{cases} u_{tt} = \Delta u, & x \in \mathbb{R}^3, \ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}^3, \\ u_t(x,0) = g(x), & x \in \Omega. \end{cases}$$

(a) Prove that

$$|u(x,t)| \le |f(x)| + \sup_{y \in B(0,1)} |g(y)|$$

for $x \in \mathbb{R}^3$ and $0 < t < 1$.

(b) Prove that

$$|u(x,t)| \le |f(x)| + \frac{3}{4\pi t^2} \int_{B(x,t)} |g(y)| \, dy + \frac{1}{4\pi t} \int_{B(x,t)} |\nabla g(y)| \, dy,$$

for $x \in \mathbb{R}^3$ and $t \ge 1$.

7.) Let $n \geq 2$, let $\Omega \subseteq \mathbb{R}^n$ be a C^∞ bounded domain, and let $T > 0$. Suppose $\vec{h} = (h_1, h_2, \ldots, h_n)$, where each component $h_j = h_j(x,t) : \overline{\Omega} \times [0,T] \to \mathbb{R}$ satisfies $h_j \in C(\overline{\Omega} \times [0,T])$. Suppose $f, g : \overline{\Omega} \to \mathbb{R}$ are continuous. Show that there is at most one function $u = u(x,t) \in C^2(\overline{\Omega} \times [0,T])$ satisfying

$$\begin{cases} u_{tt} = \Delta u + \nabla u \cdot \vec{h}, & x \in \Omega,\ 0 < t < T \\ u = 0, & x \in \partial\Omega,\ 0 \leq t \leq T, \\ u(x,0) = f(x), & x \in \Omega, \\ u_t(x,0) = g(x), & x \in \Omega. \end{cases}$$

PDE Preliminary Exam, January 2016

In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞-smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

1. Let $\Omega = \{(x,t) : x \in \mathbb{R}, t > 0\}$ and assume $u_0, v_0 \in C^1(\mathbb{R})$. Suppose $u, v \in C^1(\overline{\Omega})$ solve the system

$$u_t + u_x = u \quad \text{on} \quad \overline{\Omega},$$

$$v_t + v_x = -v + u \quad \text{on} \quad \overline{\Omega},$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad x \in \mathbb{R}.$$

Find $u(x,t)$, $v(x,t)$ in terms of u_0, v_0.

2. Let $R > 0$. Assume $u \in C^2(\overline{B_R(0)})$ is nonnegative and satisfies $u(0) = 0$,

$$0 \le \Delta u \le 1 \quad \text{on} \quad \mathrm{B_R}(0).$$

Let u_1, u_2 be the solutions of the following problems

$$\Delta u_1 = \Delta u \quad \text{on} \quad \mathrm{B_R}(0),$$

$$u_1 = 0 \quad \text{on} \quad \partial\mathrm{B_R}(0).$$

$$\Delta u_2 = 0 \quad \text{on} \quad \mathrm{B_R}(0),$$

$$u_2 = u \quad \text{on} \quad \partial\mathrm{B_R}(0).$$

(a) Prove that $u = u_1 + u_2$ on $B_R(0)$ and $u_1 \le 0, u_2 \ge 0$ on $B_R(0)$.
(b) Prove that $|u_1(x)| \le \frac{R^2}{2n}$ for all $x \in B_R(0)$. Hint: Compare u_1 with $\phi(x) = \frac{1}{2n}(R^2 - |x|^2)$.
(c) Prove that $u_2(x) \le \frac{2^{n-1}}{n}R^2$ for all $x \in B_{R/2}(0)$. Conclude $|u(x)| \le \frac{1+2^n}{2n}R^2$ for all $x \in B_{R/2}(0)$.

3. Let $n \ge 3, f \in C_0^\infty(\mathbb{R}^n)$. Assume $u \in C^\infty(\mathbb{R}^n)$ is a solution of

$$-\Delta u = f \quad \text{on} \quad \mathbb{R}^n$$

and $u(x) \to 0$ as $|x| \to \infty$. Prove there exists $C > 0$ such that

$$|u(x)| \le \frac{C}{|x|^{n-2}}$$

for all $x \in \mathbb{R}^n, x \neq 0$.

4. Let $T > 0$ and assume ϕ, h, f, g are C^∞- smooth functions. Suppose $u, v \in C^2(\overline{\Omega}_T)$ satisfy

$$\begin{aligned} u_t - \Delta u = \phi \ \text{ on } \ \Omega_{\mathrm{T}}, \\ u = h \ \text{ on } \ \partial\Omega \times (0, \mathrm{T}], \\ u = f \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\}, \end{aligned}$$

$$\begin{aligned} v_t - \Delta v = \phi \ \text{ on } \ \Omega_{\mathrm{T}}, \\ v = h \ \text{ on } \ \partial\Omega \times (0, \mathrm{T}], \\ v = g \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\}. \end{aligned}$$

Prove that $\int_\Omega |u(x,t) - v(x,t)|^2 dx \leq \int_\Omega |f(x) - g(x)|^2 dx$ for all $t \in [0, T]$.

5. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, bounded and $\int_{\mathbb{R}^n} |f| dx < \infty$. Show there exists a unique solution $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ of

$$\begin{cases} u_t = \Delta u - 2u, & \text{on } \ \mathbb{R}^n \times (0,\infty), \\ u = f, & \text{on } \ \mathbb{R}^n \times \{t = 0\}, \\ |u(x,t)| \leq Ce^{-2t}(1+t)^{-\frac{n}{2}}, & \text{for } \ (x,t) \in \mathbb{R}^n \times [0,\infty), \end{cases}$$

for some constant C depending on f, n but not on x, t.

6. Let $f \in C^1(\mathbb{R})$ with f' bounded on $\mathbb{R}$ and $f(0) = 0$. Suppose $\phi, \psi \in C^2(\overline{\Omega})$ and $u \in C^2(\overline{\Omega}_T)$ is a solution of

$$\begin{aligned} u_{tt} - \Delta u = f(u) \ \text{ on } \ \Omega_{\mathrm{T}}, \\ u = 0 \ \text{ on } \ \partial\Omega \times (0, \mathrm{T}], \\ u = \phi, \ \ u_t = \psi \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\}. \end{aligned}$$

(a) Denoting $E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2 + u^2) dx$, prove $E(t) \leq E(0)e^{Ct}$ for all $t \in [0, T]$, and for some constant $C > 0$.
(b) Prove the solution u is unique.

7. Let $p > n/2$. Suppose $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n \times [0,\infty))$ is a solution of

$$\begin{aligned} u_{tt} - \Delta u = 0 \ \text{ on } \ \mathbb{R}^{\mathrm{n}} \times [0,\infty), \\ u = \phi, \ \ u_t = \psi \ \text{ on } \ \mathbb{R}^{\mathrm{n}} \times \{\mathrm{t} = 0\}. \end{aligned}$$

Prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{|u_t| + |\nabla u|}{(1 + |x| + t)^p} dx \leq \frac{C}{(1+t)^{p-n/2}}$$

for all $t \geq 0$.

PDE Preliminary Exam, August 2015

In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞-smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

1. Let $\Omega = \{(x,t) : x \in \mathbb{R}, t > 0\}$, $b \in \mathbb{R}$ and assume $a \in C^1(\overline{\Omega}), \phi \in C^1(\mathbb{R})$ are bounded. Suppose $u \in C^1(\overline{\Omega})$ is a solution of

$$u_t + a(x,t)u_x + bu = 0 \quad \text{on} \quad \Omega,$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}.$$

(a) Prove $\sup\limits_{x\in\mathbb{R}} |u(x,t)| \leq e^{-bt} \sup\limits_{\mathbb{R}} |\phi|$ for all $t \geq 0$.
(b) Find the solution when $a = a(t)$.

2. Let $\Omega \subset \mathbb{R}^2$ and suppose $g \in C^0(\partial\Omega)$. Show that there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\Delta u + u_x - u_y = u^3 \quad \text{on} \quad \Omega,$$

$$u = g \quad \text{on} \quad \partial\Omega.$$

3. Let $\Omega \subset \mathbb{R}^n$. A function $v \in C^0(\Omega)$ is subharmonic on Ω iff for every $x \in \Omega$, there exists $r(x) > 0$ such that v satisfies the *mean-value property*:

$$v(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} v(\xi) dS(\xi)$$

for all $r \in (0, r(x)]$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.
(a) Suppose $u, v \in C^0(\Omega)$, u is harmonic on Ω, v is subharmonic on Ω, $v \leq u$ on $\partial\Omega$. Prove $v \leq u$ on Ω. You can assume the maximum principle for subharmonic functions.
(b) Let $v \in C^0(\Omega)$ be subharmonic on Ω and $B(x_0, R) \subset \Omega$. For $r \in (0, R)$ define

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi).$$

Prove g is nondecreasing on (0,R). Deduce the mean-value property

$$v(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi)$$

holds for any $\overline{B(x_0,r)} \subset \Omega$ (note, in the definition of subharmonic function, this is assumed only for sufficiently small r). Hint: for $r_1 < r_2$ use the Poisson Integral Formula on $B(x_0, r_2)$ to get a harmonic function.

4. Let $m > 0$, $T > 0$ and assume $u_0 \in C^0(\overline{\Omega})$ is nonnegative on Ω. Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ is a solution of

$$u_t = \Delta u + |\nabla u|^2 + u(m-u) \text{ on } \Omega_{\mathrm{T}},$$

$$u = 0 \text{ on } \partial\Omega \times (0, \mathrm{T}],$$

$$u = u_0 \text{ on } \Omega \times \{\mathrm{t} = 0\}.$$

Prove $0 \le u \le \max\{m, \sup_{\Omega} u_0\}$ on $\overline{\Omega}_{\mathrm{T}}$.

5. Let $1 < p < \infty$, $u_0 \in C^0(\overline{\Omega})$. Consider

$$u_t = \Delta u + |u|^{p-1}u \text{ on } \Omega_{\mathrm{T}},$$

$$u = 0 \text{ on } \partial\Omega \times (0, \mathrm{T}],$$

$$u = u_0 \text{ on } \Omega \times \{\mathrm{t} = 0\}.$$

For each u_0, let $T_{\max} = T_{\max}(u_0) \in (0,\infty]$ be the maximal time such that the problem above has a solution $u \in C^{2,1}(\overline{\Omega} \times [0, T_{\max}))$. Let $E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx$, $y(t) = \int_\Omega u^2 dx$ for $t \in [0, T_{\max})$.
(a) Prove $\frac{d}{dt}E(t) = -\int_\Omega u_t^2 dx$, $t \in (0, T_{\max})$.
(b) With $c = \frac{2(p-1)}{p+1}|\Omega|^{\frac{1-p}{2}}$ prove $\frac{d}{dt}y(t) \ge -4E(0) + cy(t)^{\frac{p+1}{2}}$, $t \in (0, T_{\max})$.
(c) Assume u_0 is nontrivial, $E(0) < 0$ and prove $T_{\max}(u_0) < \infty$.

6. Consider the initial-boundary value problem

$$u_{tt} - u_{xx} = -2 + \sin x \text{ on } (0,\pi) \times (0,\infty),$$

$$u = x^2 - \pi x, \quad u_t = 0 \text{ at } \mathrm{t} = 0,$$

$$u = 0 \text{ at } \mathrm{x} = 0, \pi.$$

(a) Find the steady state solution $u = f(x)$ of the differential equation and boundary conditions.
(b) Find the solution of the entire problem.

7. Suppose $a \in C^0(\mathbb{R}^n), a \ge 1$ on $\mathbb{R}^n$ and $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ is a solution of the problem

$$u_{tt} - \Delta u + a(x)u_t = 0 \text{ on } \mathbb{R}^n \times (0,\infty),$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n,$$
$$u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n.$$

Let $E(t) = \int_\Omega (u_t^2 + |\nabla u|^2)dx, \quad K(t) = \int_\Omega (uu_t + \frac{1}{2}au^2)dx, \quad t \in [0,\infty)$.
(a) Prove $\frac{d}{dt}E \leq 0, \quad \frac{d}{dt}(K+E) \leq -E,$ and $K+E \geq 0$ for all $t \geq 0$. You may assume finite speed of propagation of solutions (the support of $u(\cdot,t)$ is bounded in $\mathbb{R}^n$ for each $t \geq 0$).
(b) Prove $E(t) \leq Ct^{-1}$ for all $t > 0$. Hint: Integrate an inequality in (a).

1. In the region $R := \{(x,t) : x > 0, t > 0\}$, solve the PDE

$$u_t + t^2 u_x = 4u, \quad \text{with,} \quad u(0,t) = h(t), \quad u(x,0) = 1.$$

Find the conditions on h so that the solution is continuous on R.

2. Solve the following PDE (also state the domain of the solution)

$$x^2 u_x + xy u_y = u^3, \quad \text{and} \quad u = 1, \quad \text{on the curve} \quad y = x^2.$$

3. Let $a > 0$ and $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider the equation

$$\begin{cases} \Delta u &= 0, & \text{in } \ D, \\ u &= 1 + x^2 + 3xy, & \text{on } \ \partial D. \end{cases}$$

without solving the equation, find $u(0,0)$, $\max\limits_{\overline{D}} u$, and $\min\limits_{\overline{D}} u$.

4. Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ for $n > 2$. Let u be defined on $\overline{B}_1 \setminus \{0\}$. Assume that $u \in C(\overline{B}_1 \setminus \{0\}) \cap C^2(B_1 \setminus \{0\})$, u is harmonic in $B_1 \setminus \{0\}$, and

$$\lim_{|x| \to 0} \frac{u(x)}{|x|^{2-n}} = 0.$$

Prove that u can be extended to 0 so that $u \in C^2(B_1)$.

Hint: By using the maximum principle on $B_1 \setminus B_r$ for $0 < r < 1$, one proves that $u = v$ in $B_1 \setminus \{0\}$, where v is the solution of the equation

$$\begin{cases} \Delta v &= 0, & \text{in } \ B_1, \\ v &= u, & \text{on } \ \partial B_1. \end{cases}$$

5. Let Ω be a non-empty, smooth bounded domain in $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $|f'|$ is bounded. Consider the reaction-diffusion equation

$$\begin{cases} u_t - \Delta u + f(u) &= 0, & \text{in } \ \Omega \times (0,\infty), \\ u &= 0, & \text{on } \ \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x), & x \in \Omega. \end{cases}$$

Prove that C^2 solutions to the problem are unique.

6. Let $u_0 \in C_c^\infty(\Omega)$ for some non-empty, open, smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n > 2$. Assume also that $u_0 \geq 0$. Let $u \in C^\infty(\Omega \times [0,\infty))$ be a solution of the equation

$$\begin{cases} u_t & = \Delta u, \quad \text{in } \Omega \times (0,\infty), \\ u(\cdot,t) & = 0, \quad \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) & = u_0(\cdot), \quad \text{on } \Omega. \end{cases}$$

(a) Prove that for all $t > 0$,

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \quad \text{and} \quad \|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u_0\|^{\alpha}_{L^1(\Omega)} \|u(\cdot,t)\|^{1-\alpha}_{L^{2^*}(\Omega)},$$

where

$$\alpha = \frac{2^*-2}{2(2^*-1)}, \quad \text{for} \quad 2^* = \frac{2n}{n-2}.$$

(b) Prove that there is $C > 0$ depending on n, Ω such that

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx \leq -C\|u_0\|_{L^1(\Omega)}^{-\frac{2\alpha}{1-\alpha}} \left\{\int_\Omega u^2(x,t)dx\right\}^{\frac{1}{1-\alpha}}.$$

(c) Prove that (for some new $C = C(n,\Omega) > 0$)

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}(1+t)^{-\frac{n}{4}}, \quad t \geq 0.$$

Remark: The following inequalities maybe useful

(i) Hölder's inequality:

$$\|f\|_{L^p(\Omega)} \leq \|f\|^{\theta_1}_{L^{p_1}(\Omega)} \|f\|^{\theta_2}_{L^{p_2}(\Omega)},$$

with

$$\frac{1}{p} = \frac{\theta_1}{p_1} + \frac{\theta_2}{p_2}, \quad \theta_1 + \theta_2 = 1, \quad p, p_1, p_2 \in (1,\infty), \quad \theta_1, \theta_2 \in (0,1).$$

(ii) Sobolev - Poincaré inequality:

$$\|\varphi\|_{L^{2^*}(\Omega)} \leq C(n,\Omega)\|\nabla\varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in C^\infty(\Omega), \quad \varphi_{|\partial\Omega} = 0.$$

7. Let $c > 0$ be a fixed number. Solve the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + \cos(ct)\cos(x), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = x, \quad u_t(x,0) = \sin(x), & -\infty < x < \infty. \end{cases}$$

8. Let $u(x,t)$ be a C^2, compactly supported solution to the equation

$$u_{tt} - \Delta u = 0, \quad u(x,0) = 0, \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}^3 \quad t > 0.$$

Assume that $\int_{\mathbb{R}^3} g(x)^2 dx < \infty$. Show that

$$\int_0^\infty u(0,t)^2 dt \leq \frac{1}{4\pi}\int_{\mathbb{R}^3} g(x)^2 dx.$$

PDE Qualifying Exams **August 2014**

1. Let g be a given smooth function on $\mathbb{R}$. Solve the PDE

$$\begin{cases} u_x + u_y &= u^2, \qquad \text{on } \{(x,y) \in \mathbb{R}^2,\ y > 0\}, \\ u(x,0) &= g(x), \qquad x \in \mathbb{R}. \end{cases}$$

2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let u be a harmonic function in Ω and $x_0 \in \Omega$. Prove that

$$\left|\frac{\partial u(x_0)}{\partial x_i}\right| \le \frac{n}{d} \sup_{x \in \Omega} \Big|u(x) - u(x_0)\Big|, \quad \text{where} \quad d = \operatorname{dist}(x_0, \partial\Omega), \quad \forall\, i = 1, 2, \cdots, n.$$

Assume in addition that $u \ge 0$ in Ω, show that

$$\left|\frac{\partial u(x_0)}{\partial x_i}\right| \le \frac{n}{d} u(x_0), \quad \forall\, i = 1, 2, \cdots, n.$$

3. Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$, where $B_1(0)$ is an open unit ball in Ω. Let u be a harmonic function in Ω such that $u(x) \to 0$ as $|x| \to \infty$. Prove that there exist $r_0 > 1$ and $M > 0$ such that

$$|u(x)| \le \frac{M}{|x|}, \quad |u_{x_k}(x)| \le \frac{M}{|x|^2}, \quad \forall |x| \ge r_0, \quad \forall\, k = 1, 2, 3.$$

4. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let $\Omega_T = \Omega \times (0, T]$ and $u \in C^2(\overline{\Omega}_T)$ be a solution of the equation

$$\begin{cases} u_t - \Delta u + c(x,t)u &= u^2(1-u), \qquad \text{in } \Omega_T, \\ u + \frac{\partial u}{\partial \nu} &= 0, \qquad \partial\Omega \times (0, T], \\ u(x,0) &= g(x), \qquad x \in \Omega, \end{cases}$$

with some given function $c(x,t)$ and $g(x)$. Assume that $c > 0$ on $\overline{\Omega}_T$ and $0 \le g \le 1$ on $\overline{\Omega}$. Prove that $0 \le u \le 1$ on $\overline{\Omega}_T$.

5. Consider $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$ for some fixed $a > 0, b > 0$.

   (a) Use separation of variables to find the first (i.e. the smallest) eigenvalue λ_1 and eigenfunction ϕ_1 of the eigenvalue problem

$$\begin{cases} -\Delta\phi &= \lambda\phi, \qquad \Omega, \\ \phi &= 0, \qquad \partial\Omega. \end{cases}$$

   Remark: Eigenfunctions must be non-trivial.

   (b) Let g be a smooth function on $\overline{\Omega}$ and g vanishes on $\partial\Omega$. Also, let $\kappa < \lambda_1$. Assume that u is a solution of the heat equation

$$\begin{cases} u_t &= \Delta u + \kappa u, \qquad x \in \Omega,\ t > 0, \\ u(x,t) &= 0 \qquad ,\ x \in \partial\Omega,\ t > 0, \\ u(x,0) &= g(x), \qquad x \in \Omega. \end{cases}$$

   prove that $u(x,t) \to 0$ uniformly in x as $t \to \infty$.

6. Let $T \in (0,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let us denote $\Omega_T = \Omega \times (0,T)$ and Γ_T the parabolic boundary of Ω_T. Suppose that $u \in C(\overline{\Omega}_T) \cap C^2(\Omega_T)$ satisfies the PDE

$$u_t - \Delta u = c(x,t)u, \quad (x,t) \in \Omega_T$$

for some $c \in C(\overline{\Omega}_T)$ and $c \leq 0$. Show that if $u \geq 0$ on Γ_T, then

$$\max_{(x,t)\in\overline{\Omega}_T} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

Give a counter example showing that the conclusion does not hold if the condition $u \geq 0$ on Γ_T is violated.

7. Let $T \in (0,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. Suppose that $u \in C^2(\overline{\Omega} \times [0,T])$ is a classical solution of the equation

$$\begin{cases} u_{tt} - \Delta u &= f(x,t), & \Omega \times (0,T), \\ u(x,t) &= 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Let

$$E(t) = \frac{1}{2}\int_\Omega \Big[u_t^2(x,t) + |\nabla u|^2(x,t)\Big]dx$$

(a) Prove that

$$E(t) \leq e^T\Big[E(0) + \frac{1}{2}\int_0^T\int_\Omega f^2(x,s)dxds\Big], \quad \forall t \in [0,T].$$

(b) Use the energy estimate to prove the uniqueness of the classical solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u &= f(x,t), & \Omega \times (0,T), \\ u(x,t) &= 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) &= g(x), & x \in \Omega, \\ u_t(x,0) &= h(x), & x \in \Omega. \end{cases}$$

8. Let $f \in C^1(\mathbb{R}^3)$ with compact support. Suppose that $u \in C^2(\mathbb{R}^3 \times (0,\infty))$ and u solves the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u &= 0, & \mathbb{R}^3 \times (0,\infty), \\ u(x,0) &= 0, & x \in \mathbb{R}^3, \\ u_t(x,0) &= f(x), & x \in \mathbb{R}^3. \end{cases}$$

Prove that there is $M > 0$ such that

$$|u(x,t)| \leq \frac{M}{1+t}\Big[\|f\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)} + \|\nabla f\|_{L^1(\mathbb{R}^3)}\Big], \quad \forall t \geq 0.$$

PDE Qualifying Exam Spring 2014

1.) (a) Solve the following Cauchy problem on $\mathbb{R}^2$:

$$\begin{cases} u_x + u_y = x + y \\ u = x^3 \text{ on the line } y = -x. \end{cases}$$

(b) For what C^1 function or functions $f(x)$ does the Cauchy problem on $\mathbb{R}^2$:

$$\begin{cases} u_x + u_y = 3u \\ u = f(x) \text{ on the line } y = x \end{cases}$$

have a solution? Prove your answer.

2.) Consider Burger's equation

$$(*) \qquad \begin{cases} uu_x + u_y = 0, & \text{for } x \in \mathbb{R}, y > 0 \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

with initial data

$$f(x) = \begin{cases} 4, & \text{for } x < 0, \\ 4 - \frac{x}{2}, & \text{for } 0 \le x \le 2, \\ 3, & \text{for } x > 2. \end{cases}$$

(a) Find, with proof, the smallest $y^* > 0$ such that a shock occurs at (x, y^*) for some $x \in \mathbb{R}$.

(b) Find $u(x,y)$ satisfying $(*)$ for $x \in \mathbb{R}$ and $0 \le y < y^*$, except on two line segments where the partial derivatives of u may not exist.

(c) Find the integral, or weak, solution $u(x,y)$ of $(*)$ for $y \ge 0$.

3.) (a) Suppose $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(x) > 0$ for all $x \in \mathbb{R}^n$. Suppose $u \in C^2(\mathbb{R}^n)$ satisfies

$$\Delta u - f(x)u = 0$$

on $\mathbb{R}^n$, and $u(x) \to 0$ uniformly as $|x| \to \infty$. Prove that u is identically 0.

(b) Find a non-trivial solution of $\Delta u + u = 0$ in $\mathbb{R}^3$ such that $u(x) \to 0$ uniformly as $|x| \to \infty$. Hint: look for a radial solution $u(x,y,z) = v(r)$ where $r = \sqrt{x^2+y^2+z^2}$ and note that $rv'' + 2v' = (rv)''$.

4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence of harmonic functions on Ω such that

$$\int_{\Omega} |u_n(x) - u_m(x)|^2 dx \longrightarrow 0$$

as $\max\{n, m\} \to \infty$. Prove that u_n converges to a harmonic function on Ω.

5.) Suppose $u = u(x,t) \in C^2([0,1] \times [0,T])$ satisfies

$$\begin{cases} u_t = u_{xx} + tu_x, & x \in [0,1], t \in [0,T] \\ u_x(0,t) = u_x(1,t) = 0, & t \in [0,T]. \end{cases}$$

Prove that

$$\max_{[0,1]\times[0,T]} u(x,t) = \max_{[0,1]} u(x,0).$$

If you use a major theorem in PDE in your solution, provide the proof of that theorem.

6.) (a) Suppose $u = u(x,t) \in C(\mathbb{R}^n \times [0,\infty)) \cap C^2(\mathbb{R}^n \times (0,\infty))$ satisfies

$$\begin{cases} u_t = \Delta u, & \text{for } x \in \mathbb{R}^n, t > 0, \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $f(x) \geq 0$ is a C^∞, bounded function satisfying $\int_{\mathbb{R}^n} f(x)\,dx = 2$. Suppose u satisfies

$$|u(x,t)| \leq Ae^{\alpha|x|^2},$$

for some positive constants α and A. Prove that $\lim_{t\to\infty} u(x,t) = 0$ and $\int_{\mathbb{R}^n} u(x,t)\,dx = 2$ for all $t > 0$.

(b) Does there exist a bounded solution $u(x,t) \in C(\mathbb{R}^n \times [0,\infty)) \cap C^2(\mathbb{R}^n \times (0,\infty))$ of the initial value problem

$$\begin{cases} u_t = \Delta u + \frac{\cos(|x|^2+1)}{1+t^2}, & \text{for } x \in \mathbb{R}^n, t > 0, \\ u(x,0) = 0, & \text{for } x \in \mathbb{R}^n? \end{cases}$$

Justify your answer.

7.) Suppose $u = u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$ satisfies

$$\begin{cases} u_{tt} - u_{xx} + u = 0, & \text{for } x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = g(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

where f and g are C^∞ and have compact support.

(a) For any $(x_0, t_0) \in \mathbb{R} \times (0,\infty)$ and $0 \le t \le t_0$, let $I(t)$ be the interval

$$I(t) = [x_0 - t_0 + t, x_0 + t_0 - t].$$

Define

$$e(t) = \int_{I(t)} [u^2 + u_t^2 + u_x^2](x,t)\, dx,$$

for $0 \le t \le t_0$. Prove that e is non-increasing on $[0, t_0]$.

(b) Suppose that $f(x) = 0$ and $g(x) = 0$ for $|x| \ge 1$. Prove that $u(x,t) = 0$ for $|x| > t + 1$, for all $t > 0$.

8.) Suppose $u = u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$, is the solution of the wave equation

$$\begin{cases} u_{tt} = \Delta u, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Suppose g and h are C^∞ with $f(x) = g(x) = 0$ for all x such that $|x| \ge R$, for some $R > 0$. The kinetic energy is

$$k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)\, dx$$

and the potential energy is

$$p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t)\, dx.$$

(a) Prove that $k(t) + p(t)$ is constant.

(b) Prove that $k(t) = p(t)$ for all $t > R$.

PDE Qualifying Exam August 12, 2013

1.) Consider the equation

$$(*) \quad u_x + 2u_y = u,$$

for $(x, y) \in \mathbb{R}^2$.

(a) Solve $(*)$ with the Cauchy data $u(x, x) = e^{3x}$ for all $x \in \mathbb{R}$.

(b) Suppose u satisfies $(*)$ with Cauchy data $u(x, 2x) = f(x)$. Prove that $f(x) = Ce^x$ for some constant C.

(c) For each constant $C \neq 0$, show that $(*)$ with Cauchy data $u(x, 2x) = Ce^x$ has infinitely many solutions.

2.) Reduce the following equation on $\mathbb{R}^2$:

$$u_{xx} + 6x^2 u_{xy} + 9x^4 u_{yy} + 6x u_y + y - x^3 = 0$$

to canonical form and find the general solution.

3.) Let $\Omega \subseteq \mathbb{R}^n$ be a smooth (C^∞), bounded open set. Consider the problem

$$(**) \qquad \begin{cases} \Delta u(x) = f(x), & \text{for } x \in \Omega \\ u(x) + \frac{\partial u}{\partial n} = g(x), & \text{for } x \in \partial\Omega. \end{cases}$$

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$, and $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$.

(a) Prove that there is at most one $u \in C^2(\overline{\Omega})$ satisfying $(**)$.

(b) Suppose $u \in C^2(\overline{\Omega})$ satisfies $(**)$, with $f \geq 0$ on Ω and $g \leq 0$ on $\partial\Omega$. Prove that $u \leq 0$ on Ω.

4.) Suppose $u = u(x,t) \in C([0,1] \times [0, \infty)) \cap C^2((0,1) \times (0, \infty))$, and u satisfies

$$\begin{cases} u_t = u_{xx}, & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = 4x(1-x), & \text{for } 0 \leq x \leq 1. \end{cases}$$

Prove that

(a) $0 < u(x,t) < 1$ for $0 < x < 1$, $t > 0$;

(b) $u(1-x, t) = u(x,t)$ for $0 \leq x \leq 1$, $t > 0$;

(c) $-8 < u_{xx}(x,t) < 0$ for $0 < x < 1$, $t > 0$;

(d) $\int_0^1 u^2(x,t)\, dx$ is a strictly decreasing function of t.

5.) Suppose $u = u(x,t) \in C^2([0,1] \times [0,\infty))$ satisfies

$$\begin{cases} u_{tt} - u_{xx} = -\frac{u}{1+u^2}, & \text{for } 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = g(x), & \text{for } 0 \leq x \leq 1, \end{cases}$$

where g is a given function satisfying $g(0) = g(1) = 0$.

(a) Define

$$E(t) = \frac{1}{2} \int_0^1 u_t^2 + u_x^2 + \log(1+u^2)\, dx,$$

for $t \geq 0$. Prove that E is constant.

(b) Show that there exists $C > 0$ such that $|u(x,t)| \leq C$ for all $x \in [0,1]$ and $t \geq 0$.

6.) Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Suppose $u \in C^1(\overline{\Omega})$ and

$$\int_{\partial B(x,r)} \frac{\partial u}{\partial n}\, dS \geq 0$$

for every $x \in \mathbb{R}^n$ and $r > 0$ such that $B(x,r) \subseteq \Omega$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$ and dS is surface measure on $\partial\Omega$. Prove that u is subharmonic on Ω. Warning: a subharmonic function is not necessarily C^2.

(b) Prove the converse of part (a) under the additional assumption that $u \in C^2(\overline{\Omega})$.

7.) Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded open set. Let $h \leq 0$ be a continuous function on $\overline{\Omega} \times [0,\infty)$. Prove that there exists at most one function $u = u(x,t) \in C^2(\overline{\Omega} \times [0,\infty))$ satisfying

$$\begin{cases} u_t = \Delta u + h(x,t)u, & \text{for } x \in \Omega, t \geq 0 \\ u(x,0) = f(x), & \text{for } x \in \Omega, \\ u(x,t) = g(x,t), & \text{for } x \in \partial\Omega, t \geq 0. \end{cases}$$

8.) Suppose $u = u(x,t) \in C^2(\mathbb{R}^3 \times [0,\infty))$, is the solution of the wave equation

$$\begin{cases} u_{tt} = \Delta u, & \text{for } x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0, & \text{for } x \in \mathbb{R}^3, \\ u_t(x,0) = g(x), & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Suppose $g(x) = 1$ for $|x| > 1$. Prove that

$$u(x,t) = t$$

if (i) $|x| > t+1$ or (ii) $|x| < t-1$.

JANUARY 2013 PDE PRELIM

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded C^2 function that satisfies

$$\nabla f = G,$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\int_{\partial B_r(x_0)} G(x) \cdot (x - x_0) dA(x) = 0,$$

for all $x_0 \in \mathbb{R}^n$, $r > 0$. Prove that f is constant.

Problem 2. Let $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t < \infty\}$. Assume that $u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$ satisfies the initial boundary value problem given by the equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

in the interior of the region Ω, together with the boundary conditions

$$u(x,0) = f(x),\ u(0,t) = \alpha(t),\ u(1,t) = \beta(t),$$

where $f(0) = \alpha(0),\ f(1) = \beta(0)$.

(a) Show that $u(x,t)$ cannot have a maximum where $\partial^2 u/\partial^2 x < 0$ in the interior of the region in (x,t) space with $t > 0$ and $0 < x < 1$.
(b) State the strong maximum/minimum principle for the previous IVBP.
(c) Using a maximum/minimum principle show that if $f(x) \geq 0,\ \alpha(t) \geq 0$, and $\beta(t) \geq 0$, then $u(x,t) \geq 0$.

Problem 3. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and bounded and satisfies the PDE

$$u(x,y) = a(x,y)u_x(x,y) + b(x,y)u_y(x,y).$$

(a) Show that if a and b are constant functions, then u is identically 0.
(b) Prove that if $a = 1 + x^2$ and $b = 1 + y^2$, the above PDE has non-vanishing bounded solutions.

Problem 4. Consider the cube $\Omega = (1,2) \times (1,2) \times (1,2)$. Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies

$$yu_{xx} + zu_{yy} + xu_{zz} = 1$$

in Ω, with $u = 0$ on the boundary $\partial\Omega$. Prove that $u \geq -\frac{1}{8}$.

Hint. Compare with a function of the type $v(\vec{x}) = a + b|\vec{x} - \vec{x}_0|^2$, where $a, b \in \mathbb{R}$, $\vec{x}_0 \in \mathbb{R}^3$.

Problem 5. Consider the unbounded domain $\Omega = \{(x, y) : y > x^2\} \subset \mathbb{R}^2$. Suppose u is bounded and harmonic on Ω, and vanishes on $\partial\Omega$. Show $u \equiv 0$.

Hint. Test with $u\chi$, where $\chi(y)$ is a cutoff function in the second variable y, and is nonconstant only on $y \in [\ell, 2\ell]$.

Problem 6. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^3 \times [0, \infty), \\ u(x, 0) = 0 & x \in \mathbb{R}^3, \\ u_t(x, 0) = \psi(x) & x \in \mathbb{R}^3, \end{cases}$$

where $\psi \in C^\infty(R^3)$ has compact support. Let $p \in [2, \infty)$. Prove that there exists $C > 0$ such that:

(a) $|\nabla u(x,t)| \leq C(1+t)^{-1}$ for all $(x,t) \in \mathbb{R}^3 \times [0, \infty)$,
(b) $\int_{\mathbb{R}^3} |\nabla u(x,t)|^p dx \leq C(1+t)^{2-p}$ for all $t \geq 0$.

Problem 7. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = \phi(x) & x \in \mathbb{R}^n, \\ u_t(x, 0) = \psi(x) & x \in \mathbb{R}^n, \end{cases}$$

where $\phi, \psi \in C^\infty(\mathbb{R}^n)$ have compact support. Prove that there exists $C, T > 0$ such that

$$\int_{\mathbb{R}^n} \frac{(|u_t| + |\nabla u|)^4}{1 + |x| + t} dx \geq Ct^{-n-1}$$

for all $t \geq T$.

SOLUTIONS

Q1. G is C^1 since f is C^2. Using the integral condition and the divergence theorem we obtain that $\int_{\partial B} G\cdot n dA = \int_B \text{div } G = 0$ on any ball B. Since G is C^1 it follows that div $G = 0$ everywhere. Taking the divergence of the first equation we obtain div $\nabla f = \Delta f =$ div $G = 0$, i.e. f is harmonic. Since f is also bounded, it must be constant.

Q2. Will type it soon.

Q3. Along the characteristic curves $\dot{x} = a$, $\dot{y} = b$, the solution u satisfies the equation $\dot{z} = z$, hence $z(t) = z(0)e^t$. For $t \in \mathbb{R}$, this is bounded exactly if $z(0) = 0$. The reasoning with $t \in \mathbb{R}$ applies for a, b constant functions, because then the characteristic curves do exist for all t, namely $x(t) = x_0 + at$, $y(t) = y_0 + bt$. [The same reasoning would apply for any locally Lipschizt functions $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ that satisfy (eg) linear bounds $|a(x,y)| + |b(x,y)| \le C_0(|x| + |y|)$, by some ODE theory that we may not assume known in this generality, and which would guarantee global existence in time for the characteristic curves.]

In contrast, for $\dot{x} = 1 + x^2$, $\dot{y} = 1 + y^2$, we cover the plane with characteristic curves $x(t) = \tan(t+c_0) = \tan(t+\arctan x_0)$, $y(t) = \tan(t+c_1) = \tan(t+\arctan y_0)$ that exist for an interval of finite length $\le \pi$ only. We do not need $z(0) = 0$ for $z(t) = z(0)e^t$ to be bounded on this interval. Specifically, we can choose initial data $x(0) = s$, $y(0) = -s$, $z(0) = f(s)$ for any bounded function f. Then

$$u(\tan(t + \arctan s), \tan(t - \arctan s)) = f(s)e^t$$

i.e.,

$$u(x,y) = \exp\left[\frac{1}{2}(\arctan x + \arctan y)\right] f\left[\frac{1}{2}(\arctan x - \arctan y)\right]$$

Q4. We consider $v(x,y,z) := M + \frac{1}{6}\left((x-\frac{3}{2})^2 + (y-\frac{3}{2})^2 + (z-\frac{3}{2})^2\right)$ where M is yet to be determined. (It will turn out that we want $M = -\frac{1}{8}$.) We want to show, by maximum principle, that $w := u - v \ge 0$.

First we note that on Ω, it holds $yv_{xx} + zv_{yy} + xv_{zz} = \frac{2}{6}(x+y+z) > 1$. Therefore $yw_{xx} + zw_{yy} + xw_{zz} < 0$ in Ω. Now w does have a minimum on the compact $\bar{\Omega}$. If the minimum were in the interior, we'd have $w_{xx} \ge 0, w_{yy} \ge 0, w_{zz} \ge 0$ there, and thus $yw_{xx} + zw_{yy} + xw_{zz} \ge 0$ in violation of the DE. So $\min w$ is taken on at the boundary, where it equals $-\max v = -M - \frac{1}{6}\left((\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2\right) = -M - \frac{1}{8}$, which equals 0 for our choice $M = -\frac{1}{8}$.

So we have $w \ge 0$, i.e., $u \ge v \ge M = -\frac{1}{8}$ on $\bar{\Omega}$.

Q5. We can design χ in such a way that $\chi(y) = 1$ for $y \le \ell$, $\chi(y) = 0$ for $y \ge 2\ell$, $|\chi'| \le c/\ell$, $|\chi''| \le c/\ell^2$.

Then

$$0 = \int_\Omega \Delta u\,(u\chi) = -\int_\Omega \nabla u \cdot (\nabla(u\chi)) = -\int_\Omega |\nabla u|^2\chi - \frac{1}{2}\int_\Omega \nabla(u^2)\cdot\nabla\chi$$
$$= -\int_\Omega |\nabla u|^2\chi + \frac{1}{2}\int_\Omega u^2\Delta\chi - \frac{1}{2}\int_{\partial\Omega} u^2\partial_\nu\chi\, dS.$$

The boundary term vanishes; the second term, with u bounded by M, can be estimated by $M^2(c/\ell^2)(c\ell^{3/2})$, hence it goes to 0 as $\ell \to \infty$. Hence we find, in this limit, that $0 = -\int_\Omega |\nabla u|^2$, and $u \equiv const.$ By DBC, $u \equiv 0$.

Q6 & Q7. See Henry's sheet.

August

PDE Preliminary Exam, [redacted] 2012

In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞-smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0,T]$, $\Gamma_T =$ parabolic boundary of $\Omega_T = \overline{\Omega}_T \setminus \Omega_T$.

Problem 1. Let $Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 0\}$. Find the solution $u \in C^1(\Omega)$ of the initial-value problem

$$-2xu_x + (x+y)u_y = 0, \quad (x,y) \in Q,$$

$$u(x,0) = x, \quad x > 0.$$

Problem 2. Let $\Omega = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}, S = \{x \in \mathbb{R}^3 : |x| = 1\}$. Suppose $u \in C^2(\Omega) \cap C^0(\Omega \cup S)$ satisfies $\Delta u \geq 0$ on Ω, $u = 0$ on S and u is bounded on Ω. Prove $u \leq 0$ on Ω.
Hint: Consider $v(x) = u(x) - \epsilon(1/|x| - 1)$ on an appropriate subdomain of Ω.

Problem 3. Suppose $\alpha \in \mathbb{R}, T > 0$ and $f \in C^0(\overline{\Omega})$ with $f > 0$ on Ω. Let $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ be a solution of

$$u_t = \Delta u + f(x) + \alpha u \quad \text{on} \quad \Omega_{\mathrm{T}},$$

$$u = 0 \quad \text{on} \quad \Gamma_{\mathrm{T}}.$$

Prove $u \geq 0$ and $u_t \geq 0$ on $\Omega \times [0,T]$.

Problem 4. Let $a, b \in \mathbb{R}, T > 0$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega}_T)$ is a solution of

$$u_{tt} - \Delta u + au_{x_1} + bu = 0 \quad \text{on} \quad \Omega_{\mathrm{T}},$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,\mathrm{T}],$$

$$u = \phi \quad \text{on} \quad \Omega \times \{\mathrm{t} = 0\},$$

$$u_t = \psi \quad \text{on} \quad \Omega \times \{\mathrm{t} = 0\}.$$

Denoting the energy $E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)dx$, prove $E(t) \leq E(0)e^{kt}$ for all $t \in [0,T]$, for some constant $k > 0$. Here $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

Problem 5. Let $Q = \{(x,t) : x > 0, t > 0\}$. Find the solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of

$$u_{tt} - u_{xx} = 0, \quad (x,t) \in Q,$$
$$u(x,0) = x, \quad x > 0,$$
$$u_t(x,0) = -1, \quad x > 0,$$
$$u_x(0,t) + tu(0,t) = 1, \quad t > 0.$$

Problem 6. Consider the heat equation

$$u_t = \Delta u \quad \text{on} \quad \Omega_{\mathrm{T}}$$

and define $E(t) = \int_\Omega u(x,t)^2 dx, t \in [0,T]$. With Dirichlet boundary conditions $u = 0$ on $\partial\Omega \times (0,T]$, in order to prove backward uniqueness of solutions, it is sufficient to establish $E'^2 \leq EE''$ on [0,T]. Prove the same inequality for Robin boundary conditions $\partial u/\partial n = g(x)u$ on $\partial\Omega \times (0,T], g \in C^0(\partial\Omega)$.

Problem 7. Let $G(x,y)$ be the Green's function for $-\Delta$ on Ω with Dirichlet boundary conditions. Define $g(x) = \int_\Omega G(x,y)dy, x \in \overline{\Omega}$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$-\Delta u = e^{-u} \quad \text{on} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

(a) Find $-\Delta g$.
(b) Prove there exists a constant $m > 0$ such that $mg \leq u \leq g$ on Ω. Express m in some explicit form involving g.

PDE Preliminary Exam, January 2012

In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞- smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0,T]$, $\Gamma_T =$ parabolic boundary of $\Omega_T = \overline{\Omega}_T \setminus \Omega_T$.

Problem 1. Find all positive solutions u defined on all of $\mathbb{R}^2$ to the equation $xu_x + yu_y = (x^2+y^2)/u$.

Problem 2. Suppose $f \in C^0(\partial\Omega), f \geq 0$ on $\partial\Omega$. Show that if a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary-value problem

$$-\Delta u = \frac{1}{1+u^2} \quad \text{on} \quad \Omega,$$

$$u = f \quad \text{on} \quad \partial\Omega,$$

exists, then it is unique.

Problem 3. Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ is a solution of

$$u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^3 \times [0,\infty),$$

$$u(x,0) = 0, \quad x \in \mathbb{R}^3,$$

$$u_t(x,0) = g(x), \quad x \in \mathbb{R}^3,$$

where $g \in C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Prove that there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty u(x,t)^2 \, dt \leq C\|g\|^2_{L^2(\mathbb{R}^3)}.$$

Problem 4. Let $T > 0$ and suppose $f \in C^1(\mathbb{R}), f(0) = 0$. Consider the problem

$$u_t = \Delta u + f(u) \quad \text{on} \quad \Omega_T,$$

$$u = 0 \quad \text{on} \quad \Gamma_T.$$

Prove this has a solution $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ and that the solution is unique.

Problem 5. Let $\Omega = (0, \pi), Q = \Omega \times (0, \infty), f \in C^0([0, \pi]), f(0) = f(\pi) = 0$. Prove the problem

$$u_t = u_{xx} + u^2 \text{ on } \mathrm{Q},$$

$$u = 0 \text{ on } \partial\Omega \times (0, \infty),$$

$$u = f \text{ on } \Omega \times \{\mathrm{t} = 0\},$$

has no solution $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ if $I = \int_0^\pi f(x) \sin x \, dx$ is sufficiently large and positive.
Hint: Derive a differential inequality for $F(t) = \int_0^\pi u(x,t) \sin x \, dx$ and obtain a contradiction.

Problem 6. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\Delta u = u^3 - u \text{ on } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

Prove
(a) $-1 \leq u \leq 1$ on Ω,
(b) $|u(x)| \neq 1$ for all $x \in \Omega$.

Problem 7. Let $T > 0, 1 < p \leq m$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega}_T)$ is a solution of

$$u_{tt} - \Delta u + u_t |u_t|^{m-1} = u|u|^{p-1} \text{ on } \Omega_\mathrm{T},$$

$$u = 0 \text{ on } \partial\Omega \times (0, \mathrm{T}],$$

$$u = \phi \text{ on } \Omega \times \{\mathrm{t} = 0\},$$

$$u_t = \psi \text{ on } \Omega \times \{\mathrm{t} = 0\}.$$

Denote $H(t) = \frac{1}{2}\|u_t(\cdot, t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(\cdot, t)\|^2_{L^2(\Omega)} + \frac{1}{p+1}\|u(\cdot, t)\|^{p+1}_{L^{p+1}(\Omega)}$, $t \in [0, T]$ (H is not the energy for the p.d.e.). Prove that for some constant $c > 0, H(t) \leq H(0)e^{ct}$ for all $t \in [0, T]$.
Hint: Calculate $\dot{H}(t)$.

Prelim Aug 2011 Partial Differential Equations

Problem 1:

Prove that every positive harmonic function in all of $\mathbf{R}^n$ is a constant. Conclude that every semi-bounded harmonic function in all of $\mathbf{R}^n$ is a constant.

Problem 2:

Show that the damped Burger's equation $u_t + uu_x = -u$, for $x \in \mathbf{R}$, $t \geq 0$, with initial data $u(x,0) = \phi(x)$ (for a positive C^1 function ϕ) has a global solution for $t \geq 0$, provided $\phi'(x) > -1$.

Problem 3:

Let $Q = \mathbf{R}^n \times (0,\infty)$, $f \in L^1(\mathbf{R}^n)$, and let $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ be the solution of the problem

$$\begin{aligned} u_t - \Delta u + u &= 0 \qquad \text{for } t > 0, x \in \mathbf{R}^n \\ u(x,0) &= f(x) \qquad \text{for } x \in \mathbf{R}^n \,. \end{aligned}$$

subject to the growth condition $|u(x,t)| \leq Ae^{\alpha x^2}$ for $x \in \mathbf{R}^n$ and $t \geq 0$, with certain positive constants A, α. Show that

$$\|u(\cdot,t)\|_{L^\infty}(\mathbf{R}^n) \leq Ct^{-n/2}e^{-t}\|f\|_{L^1(\mathbf{R}^n)}$$

for all $t > 0$.

Problem 4:

Let $Q = \mathbf{R}^n \times (0,\infty)$, $f \in L^1(\mathbf{R}^n)$, and $g \in C^0[0,\infty) \cap L^1(0,\infty)$. Assume that $\lim_{t\to\infty} g(t)$ exists. Suppose $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ satisfies

$$\begin{aligned} u_t - \Delta u &= g(t) \qquad \text{on } Q \\ u &= f \qquad \text{on } \mathbf{R}^n \times \{t = 0\} \end{aligned}$$

and that the usual growth condition that implies uniqueness is satisfied. Show

$$\lim_{t\to\infty} u(x,t) = \int_0^\infty g(t)\,dt \text{ and } \lim_{t\to\infty} u_t(x,t) = 0$$

for each $x \in \mathbf{R}^n$.

Problem 5:

Assume in a bounded domain $\Omega \subset \mathbf{R}^n$, we have a solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ to $\Delta u = u^3 - 1$ and a solution v to $\Delta v = v - 1$, each vanishing at the boundary. Show that $0 < v \leq u \leq 1$ in Ω.

Problem 6:

Let $g \in C^2(\mathbf{R}^3)$ satisfy the conditions

$$|g(x)| < C \quad \text{and} \quad \int_{\mathbf{R}^3} |\nabla g(x)|\,dx < 4\pi C \quad \text{and} \quad \lim_{|x|\to\infty} g(x) = 0$$

and consider a classical solution u to the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \mathbf{R}^3 \times (0,\infty) \\ u(x,0) &= C && \text{for } x \in \mathbf{R}^3 \\ u_t(x,0) &= g(x) && \text{for } x \in \mathbf{R}^3\,. \end{aligned}$$

where C is a given positive constant. Prove that $u(x,t) > 0$ for all $(x,t) \in \mathbf{R}^3 \times [0,\infty)$.

Problem 7:

Suppose $\phi \in C^\infty(\mathbf{R}^n)$ and $\psi \in C^\infty(\mathbf{R}^n)$ have support contained in the ball $B(0,r)$, and that $u \in C^2(\mathbf{R}^n \times [0,\infty))$ is a solution to

$$\begin{aligned} u_{tt} - \Delta u + \tfrac{1}{1+|x|} u_t &= 0 && \text{on } \mathbf{R}^n \times (0,\infty) \\ u(x,0) &= \phi(x) && \text{for } x \in \mathbf{R}^n \\ u_t(x,0) &= \psi(x) && \text{for } x \in \mathbf{R}^n \end{aligned}$$

Define $E(t) := \frac{1}{2}\int_{\mathbf{R}^n}(u_t^2 + |\nabla u|^2)\,dx$ and $I(t) := \int_t^\infty \int_{\mathbf{R}^n} \frac{1}{1+|x|}(u_t^2 + |\nabla u|^2)\,dx\,ds$.

(a) Prove that $\int_t^\infty \int_{\mathbf{R}^n} \frac{1}{1+|x|} u_t^2(x,s)\,dx\,ds \leq E(t)$.

For your information: it can be proved that $I(t) \leq CE(t)$. You do not need to do this; only be assured of the corollary that $I(t)$ is finite.

(b) Prove that there exists a positive constant C such that $I(t) \geq CE(2t)$ for all $t \geq r$ (with the r from the support of the data). *Hints: $I(t) \geq \int_t^{2t} \ldots$. You may assume that the support of u has the same properties as solutions to the wave equation whose initial data have support in $B(0,r)$. And you may assume that $E(t)$ is non-increasing in t.*

PDE Preliminary Exam, January 2011

In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞- smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0,T]$.

Problem 1. Prove the pde $u_x + 2xu_y = (y^2 - x^2)u^2 + 1$ cannot have a solution $u \in C^1(\mathbb{R}^2)$ in the entire plane $\mathbb{R}^2$.

Problem 2. Let $a \in \mathbb{R}$. Show the problem

$$\Delta u = u^5 + a \quad \text{on} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

has at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Problem 3. Let $Q = \mathbb{R}^n \times (0,\infty)$ and suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is a solution of

$$u_t - \Delta u = 0 \quad \text{on} \quad \mathrm{Q},$$

$$u = g(x) \quad \text{on} \quad \mathrm{R}^{\mathrm{n}} \times \{\mathrm{t} = 0\},$$

satisfying the growth condition

$$|u(x,t)| \le Ae^{\alpha|x|^2}, \quad (x,t) \in Q,$$

where A, α are positive constants.

(a) Assume that $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ does not depend on a variable x_j for some fixed j. Prove that the same is true for u.

(b) Prove that if $g \in C^\infty(\mathbb{R}^n)$ is a harmonic function on $\mathbb{R}^n$, the solution u is time independent.

Problem 4. Let $\alpha, T > 0, \gamma \in \mathbb{R}$. Suppose $\phi \in C^0(\overline{\Omega})$ and $c \in C^0(\overline{\Omega}_T)$ with $c \ge \gamma$ on $\overline{\Omega}_T$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^1(\overline{\Omega}_T)$ is a solution of

$$u_t - \Delta u + c(x,t)u = 0 \quad \text{on} \quad \Omega_\mathrm{T},$$

$$u = \phi \quad \text{on} \quad \Omega \times \{\mathrm{t} = 0\},$$

$$\partial u/\partial n + \alpha u = 0 \quad \text{on} \quad \partial\Omega \times (0,\mathrm{T}].$$

Prove $|u| \le \sup\limits_{\overline{\Omega}} |\phi| \;\; e^{-\gamma t}$ on Ω_T and prove u is unique.

Problem 5. Solve explicitely the initial-boundary value problem

$$u_{tt} - 4u_{xx} = 0, \quad x > 0, \quad t > 0,$$

with initial data

$$u(x,0) = x, \quad x > 0,$$

$$u_t(x,0) = -2, \quad x > 0,$$

and boundary condition

$$u_x(0,t) + tu(0,t) = 1, \quad t > 0.$$

Problem 6. Suppose $\Omega \subset \mathbb{R}^2$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 \ \text{ on } \ \Omega.$$

Show $\inf\limits_{\overline{\Omega}} u = \inf\limits_{\partial\Omega} u$.

Problem 7. Let $T > 0, a \in \mathbb{R}$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^1(\overline{\Omega}_T)$ is a solution of

$$u_{tt} - \Delta u + au_t = 0 \ \text{ on } \ \Omega_{\mathrm{T}},$$

$$u = \phi \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\},$$

$$u_t = \psi \ \text{ on } \ \Omega \times \{\mathrm{t} = 0\},$$

$$\partial u/\partial n = 0 \ \text{ on } \ \partial\Omega \times (0, \mathrm{T}].$$

Prove that for $t \in [0,T]$ the following inequality holds $E(t) \leq E(0)e^{a_0 t}$, where $E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)dx$ and $a_0 = \max\{0, -2a\}$.

PDE Prelim Exam, August 2010

In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with C^∞- smooth boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

Problem 1. Suppose $u \in C^1(\mathbb{R}^2)$ is a solution of $yu_x - xu_y = u$ on the entire plane $\mathbb{R}^2$. Prove $u = 0$ on $\mathbb{R}^2$.

Problem 2. Suppose $f, g \in C^1(\mathbb{R})$ with $f(0) = g(0) = 0, f' > 0$ and $g' > 0$ on $\mathbb{R}\backslash\{0\}$. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of

$$\Delta u = f(u) \text{ on } \Omega,$$

$$\partial u/\partial n + g(u) = 0 \text{ on } \partial\Omega.$$

(a) Show $u = 0$ on Ω using the maximum priciple.
(b) Show $u = 0$ on Ω using the energy method.

Problem 3. Let $T > 0, c \in C^0(\overline{\Omega}_T)$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ satisfies

$$u_t - \Delta u + c(x,t)u \leq 0 \text{ on } \Omega_{\mathrm{T}},$$

$$u \leq 0 \text{ on } \Gamma_{\mathrm{T}} \ (= \overline{\Omega}_{\mathrm{T}} \backslash \Omega_{\mathrm{T}} = \text{parabolic boundary of } \Omega_{\mathrm{T}}).$$

Prove $u \leq 0$ on Ω_T.
Hint: Consider $v = ue^{-Mt}$ for a suitable constant M.

Problem 4. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^3 \times [0, \infty),$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^3,$$

$$u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3,$$

where $g \in C^2(\mathbb{R}^3)$ has compact support. Prove that there exists $C > 0$ such that
(a) $|u_t(x,t)| \leq C(1+t)^{-1}$ for all $(x,t) \in \mathbb{R}^3 \times [0, \infty)$, and

(b) $(\int_{\mathbb{R}^3} |u_t|^6 dx)^{1/6} \leq C(1+t)^{-2/3}$ for all $t \geq 0$.

Problem 5. Suppose $u \in C^2(\mathbb{R}^n)$ satisfies $\Delta u + u^2 + 2u \leq 0$ on R^n. Show that the inequality $u \geq 1$ cannot hold on all of $\mathbb{R}^n$.
Hint: Consider the auxiliary function $v(x) = \frac{3}{2n}(R^2 - |x|^2)$ on $B(0, R)$.

Problem 6. Suppose $n \leq 3$, $\phi \in C^3(\mathbb{R}^n), \psi \in C^2(\mathbb{R}^n)$ and ϕ, ψ have compact support. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$u_{tt} - \Delta u = u^3 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}^n,$$

$$u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}^n,$$

where $\int_{\mathbb{R}^n} \phi(x)^2 dx > 0$. Define the energy
$E(t) = \int_{\mathbb{R}^n} (\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4) dx$ and $F(t) = \int_{\mathbb{R}^n} u^2 dx$ for $t \geq 0$. Assume $E(0) < 0$.
(a) Prove $E(t)$ is constant in t.
(b) Find a lower bound for $||u(\cdot, t)||_{L^4(\mathbb{R}^n)}$ and prove $F''(t) \geq 6||u_t||^2_{L^2(\mathbb{R}^n)}$ for each t.
(c) Prove $(F(t)^{-\frac{1}{2}})'' \leq 0$ for all $t > 0$ (note $(F(t)^{-\frac{1}{2}})'' = -\frac{1}{2}(FF'' - \frac{3}{2}F'^2)F^{-\frac{5}{2}})$.
(d) Provided that $F'(t) > 0$ for some $t > 0$, show $F(t) \to \infty$ as $t \to t_0^-$ for some finite $t_0 > 0$.

Problem 7. Let $Q = \mathbb{R}^n \times (0, \infty), n = 2, 3$ and $f \in C^0(\overline{Q})$. Suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is a solution of

$$u_t - \Delta u = f(x, t) \quad \text{on} \quad Q,$$

$$u = 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\}.$$

Assume $\int_{\mathbb{R}^n} f(x, t)^2 dx \leq k$ for all $t \geq 0$; and that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|f| \leq C_\varepsilon e^{\varepsilon |x|^2}$ on Q. Assume $|u| \leq Ae^{a|x|^2}$ holds on Q for some constants $a, A > 0$. Show, for some $C, \alpha > 0$, $|u| \leq Ct^\alpha$ holds on Q. Give α explicitly and explain if your reasoning depends on n. Explain the purpose of $e^{a|x|^2}$.