LINEAR ALGEBRA DIAGNOSTIC, AUGUST 2022

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex, unless otherwise stated.

(1) Suppose that T and S are both linear transformations from $\mathbb{C}^6$ to $\mathbb{C}^2$. Prove that there exists a nonzero vector $v \in \mathbb{C}^6$ such that $S(v) = T(v) = \mathbf{0}$.

(2) Let T and S be linear operators on a vector space V. Prove that there exists a nonzero vector $v \in V$ such that $S(v)$ is a multiple of $T(v)$

(3) Give an example of a matrix A whose characteristic polynomial is $(z-1)(z-5)^4$ and whose minimal polynomial is $(z-1)(z-5)^2$.

(4) Let T be a linear operator on a 3-dimensional vector space V, and suppose that the eigenvalues of T are 6 and 8, and those are the only eigenvalues. Prove that there exist at least two different 2-dimensional vector subspaces $W_1, W_2 \subset V$ such that $T(W_i) = W_i$ for $i = 1, 2$.

(5) If A is an $n \times n$ matrix such that $\|Av\| \leq 4\|v\|$ for all vectors $v \in \mathbb{R}^n$, then show that the eigenvalues of $A^T A$ are all less than or equal to 16.

(6) Let T be a self-adjoint operator on an inner product space V. If there exists a non-zero vector v such that $\|T(v) - v\| > 3\|v\|$, then show that T has an eigenvalue λ such that $|\lambda - 1| > 3$.