LINEAR ALGEBRA DIAGNOSTIC, MAY 2023

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex.

(1) Suppose that S and T are operators on a vector space V such that $(ST)^3 = 0$. Show that the composition TS is not invertible.
(2) Let V and W be vector spaces of dimensions 9 and 5 respectively. Suppose that $T \colon V \to W$ is a linear map whose range is all of W. Determine the least positive integer m such that every m-dimensional subspace of V contains a non-zero vector v with $T(v) = 0$.
(3) Suppose that A is a 2023×2023 matrix whose eigenvalues are -2, 0, 2, and 3 (and these are all the eigenvalues). If A has rank 3, find the characteristic polynomial of A.
(4) Suppose that T is an operator on a 4-dimensional vector space V. The eigenvalues of T are 3 and -2. Both $T - 3I$ and $T + 2I$ have rank 3. What are the possible Jordan canonical forms of T, up to rearranging the blocks?
(5) Let T be a self-adjoint operator on a 7-dimensional inner product space V. If $T^5(v) = -v$ for all $v \in V$, then determine the trace of T.
(6) Let A be a matrix, and let A^* denote its conjugate transpose. If $AA^*A = 0$, then show that $A = 0$. (Hint: A^*A is Hermitian.)