TOPOLOGY PRELIMINARY EXAM, AUGUST 2023

Solve eight items from part 1, and eight from part 2. Justify all your claims, in as much detail as time allows. All spaces mentioned are assumed Hausdorff.

PART 1

1. Let $f : X \to Y$ be a continuous, surjective, closed map. Prove:

(i) For each $y \in Y$ and any $U \subset X$ open neighborhood of the preimage $f^{-1}(y)$, there exists $V \subset Y$ open neighborhood of y so that $f^{-1}(V) \subset U$.

(ii) Let $y \in Y$, let $U \subset X$ be an open neighborhood of $f^{-1}(y)$. Then $f(U)$ contains an open neighborhood $V \subset Y$ of y.

2. (i) Let X be a metric space. Prove: If each $f_n : X \to \mathbb{R}$ is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.

(ii) Prove: There is no sequence of polynomials converging to $\sin(1/x)$, uniformly on the open interval $(0, 1)$.

3. (i) Let $p : X \to Y$ be a surjective local homeomorphism. If X is compact, prove that p is a covering map with finite fibers.

(ii) Give an example of a surjective local homeomorphism that is not a covering map. (And explain why both facts are true for your example.)

4. (i) Define 'homotopy equivalent spaces' and 'deformation retraction', and show that if X deformation retracts to a subspace $A \subset X$, then X and A are homotopy equivalent.

(ii) Show that the figure-eight space X and the theta-space Y are homotopy equivalent, and that they are not homeomorphic.

$$X = \{(x, y) \in \mathbb{R}^2; x^2 + (y - 1)^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2; x^2 + (y + 1)^2 = 1\}.$$

$$Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2; -1 \leq x \leq 1\}.$$

5. (i) Prove that if X is a compact connected manifold with finite fundamental group, any continuous map $f : X \to T^n$ (the n-dimensional torus) is homotopic to a constant.

(ii) Let M be the compact orientable surface of genus 2. Prove there exists $f : M \to S^1$ continuous, not homotopic to a constant.

Hint: first show that M retracts to the figure-eight space.

PART 2

6. (i) Prove: The space of rank one 2×2 real matrices (that is, nonzero matrices with determinant zero) is a 3-dimensional submanifold V of $M_{2\times 2}(\mathbb{R}) = \mathbb{R}^4$.

(ii) Find the tangent space $T_A V$ of V at the matrix $A \in V$ given, expressed as a subspace of $M_{2\times 2}(\mathbb{R})$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

7. (i) Prove: If X is a compact connected manifold with boundary, there is no smooth map $X \to \partial X$ that is the identity on ∂X (that is, ∂X is not a smooth retract of X.)

Hint: Sard's theorem.

(ii) Let X be an oriented manifold, $f : X \to \mathbb{R}$ a smooth function with $0 \in \mathbb{R}$ as a regular value. Prove that the submanifold $Z = f^{-1}(0)$ of X is orientable.

8. (i) Denote by B the closed unit ball in $\mathbb{R}^n$, with boundary S^{n-1}. Let $f : B \to \mathbb{R}^n$ be a continuous map, such that $f(S^{n-1}) \subset B$. Prove that f has a fixed point in B.

Hint: Consider $p \circ f$, where $p : \mathbb{R}^n \to B$ is the continuous map defined as the identity on B, and as the nearest-point projection to S^{n-1} on $\mathbb{R}^n \setminus B$.

(ii) Let $f : S^n \to S^n$ be a continuous map such that $||f(x) - x|| < 2$ for all $x \in S^n$ (for the usual norm in $\mathbb{R}^{n+1}$). Prove that f is surjective.

9. (i) Show that the degree of the antipodal map $\alpha : S^n \to S^n$, $\alpha(x) = -x$, is $(-1)^{n+1}$.

(ii) Prove: any (smooth) map $S^n \to S^n$ with degree different from $(-1)^{n+1}$ must have a fixed point.

10. (i) Let X be a compact smooth manifold. Prove: given any continuous map $f : X \to S^n \subset \mathbb{R}^{n+1}$ and any $\epsilon > 0$, there exists $g : X \to S^n$ smooth so that $\sup_{x\in X} ||f(x) - g(x)|| < \epsilon$.

Hint. Explain why we may approximate f by a smooth map $h : X \to \mathbb{R}^{n+1}$. Then show the normalization of h (taking values on S^n) is still close to f.

(ii) Prove that if ϵ is small enough, the approximation g in (i) is homotopic to f.

Hint: Consider a retraction $r : N_\epsilon \to S^n$ from a tubular neighborhood N_ϵ of S^n in $\mathbb{R}^{n+1}$.

Topology prelim January 2023

Do 4 Problems from each part.

PART I

Problem 1.

(i) Let $f : M \to N$ be a continuous map between non-compact manifolds. Show that if f is proper (def: preimage of compact is compact), then f is a closed map.

(ii) Prove that if $f : M \to R^n$ is a smooth injective immersion and $\phi : M \to R_+$ is a smooth proper function, then $g(x) = (f(x), \phi(x))$ defines a smooth immersion from M to R^{n+1}, which in addition is a proper map.

Problem 2.

Let $f : X \to Y$ be a function between two topological spaces and assume that X is metrizable.

(a) Show that if $A \subset X$ ant $x \in \overline{A}$, then there exists a sequence $x_n \in A$ such that $x_n \to x$.

(b) If for any convergent sequence $x_n \to x \in X$, the sequence $f(x_n)$ converges to $f(x)$, then f is continuous.

Problem 3.

Let M be a smooth manifold, $f : M \to \mathbb{R}^k$ be a C^1 map, and $N \subset \mathbb{R}^k$ a submanifold of codimension strictly greater than $\dim M$. Show that for almost every $v \in \mathbb{R}^k$ (i.e., except of a set of v of measure zero) the translated image $f(M) + v$ has empty intersection with N.

Problem 4.

(i) Let $\mathbb{Q} = \{r_1, r_2, \dots\}$ be an enumeration of the rationals, and let $A_{i_j} = (r_i - \frac{1}{2^{i+j}}, r_i + \frac{1}{2^{i+j}})$, for each $i, j \geq 1$. Show that the set

$$A = \bigcap_{j \geq 1} \bigcup_{i \geq 1} A_{ij}$$

is a residual set (countable intersection of open dense sets) and has measure zero.

(ii) A subset of $\mathbb{R}$ is *of first category* if it is contained in a countable union of closed sets with empty interior. Prove that any set S of real numbers may be written as the disjoint union of a set of measure zero and a set of first category. (*Hint:* part (i).)

Problem 5.

Let X be a non-compact manifold, $X = \cup_{i \geq 1} X_i$ a compact exhaustion(i.e. $X_i \subset X$ compact and $X_i \subset \mathrm{int}(X_{i+1})$). Suppose (Y, d) is a bounded, complete metric space, with $d(y, y') < 1$ for all $y, y' \in Y$. Consider the metric on the space of all functions $f : X \to Y$:

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in X_i} d(f(x), g(x)) \,.$$

Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(X, Y)$ and $f : X \to Y$. Show that $f_n \to f$ uniformly on compact sets if and only if $\rho(f_n, f) \to 0$.

Problem 6.

Let (Y, d) be a bounded metric space ($d(y, y') < 1$, for all $y, y' \in Y$), and $\mathcal{F}$ be a totally bounded subset in $\mathcal{C}(X, Y)$ (the space of continuous functions from X to Y) with respect to the uniform metric $d_u(f, g) := \sup_{x \in X} d(f(x), g(x))$. Then $\mathcal{F}$ is equicontinuous.

Hint: Recall that $\mathcal{F}$ totally bounded means that for any $\epsilon > 0$ there exist a finite number of functions $f_i \in \mathcal{C}(X, Y)$, $i = 1, \dots, n$, such that $\mathcal{F} \subset \cup_{i=1}^n B^{d_u}(f_i, \epsilon)$, where

$$B^{d_u}(f_i, \epsilon) = \{g \in \mathcal{C}(X, Y) : d_u(f_i, g) < \epsilon\}$$

.

PART II

Problem 1. Let X be the union of k distinct lines passing through the origin in $\mathbb{R}^3$. Compute the fundamental group of $\mathbb{R}^3 \setminus X$.

Problem 2.

(i) Let $p : X \to Y$ be a covering map. If X is path connected and Y is simply connected show that p has to be a homeomorphism.

(ii) Prove that every continuous map from S^2 to T^2 is nullhomotopic.

Problem 3.

(i) If $f : S^1 \to S^n$ is continuous, the Stone-Weierstrass theorem implies that, for any $\epsilon > 0$, there exists $h : S^1 \to R^{n+1}$ smooth, so that $|h(x) - f(x)| < \epsilon$, for all $x \in S^1$. Prove that, for ϵ sufficiently small, the map $g(x) = \frac{h(x)}{|h(x)|}$ is a smooth map to S^n, freely homotopic to f.

(ii) Using Sard's theorem and part (i), prove that for any $n > 1$ the sphere S^n is simply connected.

Problem 4.

(i) Let v be a tangent vector field on $S^n \subset R^{n+1}$. (That is, $\langle v(x), x\rangle = 0$, for all $x \in S^n$.) Show that if $v(x) \neq 0$ for all x, the map of S^n:

$$f(x) = \frac{x + v(x)}{|x + v(x)|}$$

is smooth, homotopic to the identity, and has no fixed points.

(ii) Show that the degree of the antipodal map of S^n is $(-1)^{n+1}$. Explain why this implies the situation in part (i) can only happen if n is odd. (*Hint:* See also problem 5(a)).

Problem 5.

(a) Show that if f has no fixed points, then f is homotopic to the antipodal map.

(b) Let $f : S^n \to S^n$ be a continuous map such that $\|f(x) - x\| < 1$ for all $x \in S^n$. Prove that f is surjective.

Problem 6.

For X a compact manifold, Y a manifold with $dim(Y) > 1$, $Z \subset Y$ a submanifold of codimension equal to $dim(X)$ and $f : X \to Y$ a smooth map, denote by $I_2(f, Z)$ the mod 2 intersection index of f with Z.

(i) If Y is contractible, show that $I_2(f, Z) = 0$.

(ii) If $Y = S^k$, show that $I_2(f, Z) = 0$.

Topology prelim August 2022

Do 4 Problems out of each part.

PART 1:

Problem 1.
Let X be a regular topological space (we assume that single points are closed sets). Show that for any $x \in X$ and open neighborhood U of x, there exists an open neighborhood V of x such that $\overline{V} \subset U$. Deduce that every pair of points of X have neighborhoods whose closures are disjoint.

Problem 2.

(i) Consider the set of functions $f : \mathbb{R} \to \mathbb{R}$ with the topology of pointwise convergence. Show that it is not metrizable.

(ii) Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a metric space Y. If (f_n) converges uniformly to f, prove that f is continuous.

Problem 3.
Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Prove that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i, $i = 1, 2, 3$, into two pieces of equal measure.

Hint: Use the version of Borsuk-Ulam theorem: For $f : S^2 \to \mathbb{R}^2$ continuous, there exists $x \in S^2$ such that $f(x) = f(-x)$. It will be helpful, for each $\nu \in S^2$, to define three planes orthogonal to ν that split the A_i's into equal pieces.

Problem 4.

(a) Let $p : X \to Y$ be a covering map. If X is path connected and Y is simply connected show that p has to be a homeomorphism.

(b) Prove that every continuous map from S^2 to T^2 is nullhomotopic.

Problem 5.
Consider a function $f : \mathbb{R} \to \mathbb{R}$ and let D be the set of points at which f is continuous.

(i) Show that D has to be a G_δ set (countable intersection of open sets).

(ii) Show that D cannot be a countable dense subset of $\mathbb{R}$.

Problem 6: A space X is said to be contractible if the identity map of X is homotopic to a constant map.

(a) Show that any convex open set in $\mathbb{R}^n$ is contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then all maps $f : X \to Y$ are homotopic to one another.

(d) Show that if X is contractible and Y is path connected, then all maps $f : X \to Y$ are homotopic to one another.

PART 2:

Problem 1.

(a) Let X and Y be two submanifolds of $\mathbb{R}^n$. Show that for almost every $a \in \mathbb{R}^n$ $X + a$ intersects Y transversely.

(b) Let Z be the preimage of a regular value $y \in Y$ under the smooth map $f : X \to Y$. Show that the kernel of $df_x : T_x(X) \to T_y(Y)$ at any $x \in Z$ is precisely $T_x(Z)$.

Problem 2.

(a) Let $f : X \to Y$ be a submersion and U an open set of X. Show that $f(U)$ is open in Y.

(b) If X is compact and Y is connected, show that every non trivial submersion $f : X \to Y$ is surjective.

Problem 3

Let X be a compact manifold with boundary. Show that there is no smooth map $f : X \to \partial X$ which is the identity on the boundary ∂X (i.e. $\partial f = \mathrm{id}$).

Problem 4

(a) Let $\Sigma_c = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = c\}$. Show that if $c \neq 0$ then Σ is a manifold. Show that if $c = 0$ then Σ_0 is not a manifold.

(b) Show that $\{(x, y, z) : (x - 1)^2 + y^2 + z^2 = 1\} \cap \Sigma_1$ is a 1-manifold.

Problem 5: If S^k has a vector field with no zeros then the antipodal map is homotopic to the identity.

Problem 6:

(a) Let Y be a contractible manifold with $\dim Y \geq 1$. Show that $I_2(f, Z) = 0$ for all closed proper submanifolds $Z \subset Y$ (proper meaning that Z is not identically equal to Y) and $f : X \to Y$ with X a compact and satisfying $\dim X + \dim Z = \dim Y$.

(b) Let X be a compact manifold with $\dim X \in (0, k)$, $k \geq 2$, and consider a smooth map $f : X \to S^k$. Show that for all closed submanifolds $Z \subset S^k$ such that $\dim X + \dim Z = k$ we have $I_2(f, Z) = 0$.

TOPOLOGY PRELIMINARY EXAM- JANUARY 2022

Instructions. Solve eight of the ten problems proposed, including as much detail as time allows.

PART ONE

1. *Definition:* A metric space (X, d) is *proper* if it has the Heine-Borel property (bounded sets are precompact.)

(i) Let (X, d) be a proper metric space. Prove that X is complete, locally compact and σ-compact.

(ii) Show that a metric space (X, d) is proper if and only if the distance function to a point $x \mapsto d(x, x_0)$ is a proper function on X. (Recall a continuous map is *proper* if the preimage of compact sets is compact)

2. (i) Prove: A connected, locally path connected space X is path connected. In particular, connected open sets in $\mathbb{R}^n$ are path connected.

(ii) Prove: Any collection $\mathcal{U}$ of pairwise disjoint open subsets of $\mathbb{R}^n$ is countable.

3. Let the group G act by homeomorphisms on the Hausdorff space X, with $\sim$ the orbit equivalence relation: $x \sim y \Leftrightarrow (\exists g \in G)(y = gx)$. Let $\pi : X \to Y$ be the quotient projection onto the space $Y = X/\sim$.

(i) Show that if Y is given the quotient topology, π is an open map.

(ii) Let $\Gamma = \{(x, y) \in X \times X; x \sim y\}$ be the graph of $\sim$. Show that if Γ is a closed subset of $X \times X$, then Y is Hausdorff (with the quotient topology.)

4. Let (Y, d_Y) be a complete metric space, X a set, Y^X the set of all functions $f : X \to Y$. The uniform metric on Y^X associated to d_Y is defined as:

$$d(f, g) = \sup_X \min\{d_Y(f(x), g(x)), 1\}.$$

(i) Show that (Y^X, d) is complete.

(ii) If X is a topological space, (Y, d_Y) a complete metric space, show that the set $C(X, Y)$ of continuous functions, with the uniform metric, is a complete metric space.

5. (i) Prove: If each $f_n : X \to E$ (X metric space, E Banach space) is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.

(ii) Use (i) to show that there is no sequence of polynomials converging to $1/x$ uniformly on the open interval $(0, 1)$ (justify each claim in the argument).

PART TWO

1. Let $p : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree d in k variables:

$$p(tx) = t^d p(x); \quad t \in \mathbb{R}, x \in \mathbb{R}^k.$$

(i) Prove that if $a \neq 0$ the set $M_a = \{x \in \mathbb{R}^k; p(x) = a\}$ (if non-empty) is a smooth hypersurface in $\mathbb{R}^k$ (codimension 1 submanifold) .

Hint: use the Euler identity:

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = dp$$

to show any $a \neq 0$ is a regular value of p.

(ii) Prove that all M_a with $a > 0$ are diffeomorphic to one another.

2. (i) Let V be a finite-dimensional real vector space, $T \in \mathcal{L}(V)$, $\Delta \subset V \times V$ the diagonal subspace, $\Gamma_T \subset V \times V$ the graph subspace of T. Then:

$$\Gamma_T \pitchfork \Delta \Leftrightarrow 1 \text{ is not an eigenvalue of } T.$$

($\pitchfork$ is the notation for 'transversal'.)

(ii) In this case, what is the dimension of the intersection subspace $\Gamma_T \cap \Delta$? (Justify.)

3. (i) Define 'orientable manifold'.

(ii) Let X be an oriented manifold, $f : X \to \mathbb{R}$ a smooth function with $0 \in \mathbb{R}$ as a regular value. Prove that the submanifold $Z = f^{-1}(0)$ of X is orientable.

4. (i) Prove that every continuous map from RP^2 to T^2 is nullhomotopic.

(ii) Let M be the compact orientable surface of genus 2. Prove there exists $f : M \to S^1$ continuous, not homotopic to a constant.

Hint: To begin the argument, show (with the help of clear sketches) that M admits a continuous retraction to $S^1 \vee S^1$, the 'figure eight' space.

5. (i) Show that if $h : S^n \to S^n$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

(ii) Let $f : S^n \to S^n$ be a continuous map such that $||f(x) - x|| < 1$ for all $x \in S^n$ (for the usual norm in $\mathbb{R}^{n+1}$.) Prove that f is surjective.

Topology Preliminary Examination–August 2021

Instructions: solve 8 of the 10 problems given. For a passing grade, at least 6 problems must be given correct and complete solutions; including at least 2 from part I and 2 from part II.

PART I

1. Let the group G act by homeomorphisms on the Hausdorff space X, with $\sim$ the orbit equivalence relation: $x \sim y \leftrightarrow (\exists g \in G)(y = gx)$. Let $\pi : X \to Y$ be the quotient projection onto the space $Y = X/\sim$.

(i) Show that if Y is given the quotient topology, π is an open map.

(ii) Let $\Gamma = \{(x, y) \in X \times X; x \sim y\}$ be the graph of $\sim$. Show that if Γ is a closed subset of $X \times X$, then Y is Hausdorff (with the quotient topology.)

2. (i) Prove: A separable metric space cannot contain an uncountable discrete set.

(ii) Show that $C(\mathbb{R}; [0, 1])$ is not separable (with the uniform metric, $d(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$).

3. *Definition:* A metric space (X, d) is *proper* if it has the Heine-Borel property (bounded sets are precompact.)

(i) Let (X, d) be a proper metric space. Prove that X is complete, locally compact and σ-compact.

(ii) Show that a metric space (X, d) is proper if and only if the distance function to a point $x \mapsto d(x, x_0)$ is a proper function on X.

4. Let X be locally compact Hausdorff and σ-compact, with compact exhaustion $(K_n)_{n\geq 1}$. Define a metric on $C(X)$ (real-valued continuous functions on X) by:

$$\rho(f, g) = \sum_{n=1}^{\infty} \rho_n(f, g), \quad \rho_n(f, g) = \min\{\frac{1}{2^n}, \sup_{x\in K_n} |f(x) - g(x)|\}.$$

Show that the topology induced by ρ on $C(X)$ is equivalent to the topology of uniform convergence on compact sets.

5. *Definition:* A family $\mathcal{F}$ of maps $f : \mathbb{R}^n \to \mathbb{R}^k$ is a *locally Lipschitz family* if for all $R > 0$ we may find $L > 0$ (depending on R) so that, for all $f \in \mathcal{F}$:

$$||x|| \leq R, ||y|| \leq R \Rightarrow ||f(x) - f(y)|| \leq L||x - y||.$$

Let $\mathcal{F}$ be a locally Lipschitz family of maps $f \in C(\mathbb{R}^n; \mathbb{R}^k)$, which is also bounded at each point ($||f(x)|| \leq M(x)$ for all $f \in \mathcal{F}$, with $M(x) > 0$ depending on x, but not on f.) Show that any sequence $f_n \in \mathcal{F}$ admits a subsequence converging uniformly on compact sets to a map $g \in C(\mathbb{R}^n; \mathbb{R}^k)$.

(You may assume the Arzelà-Ascoli theorem for maps from compact spaces.)

PART II

6. Let $X \subset \mathbb{R}^N$ be a compact smooth embedded submanifold (of dimension $m < N$). Prove that every continuous map $f : X \to S^n \subset \mathbb{R}^{n+1}$ may be approximated by a smooth map, homotopic to f. That is, for any $\epsilon > 0$ there exists $\hat{g} : X \to S^n$ smooth, homotopic to f, and ϵ-close to f in the sup distance. (Include the proof that the maps are homotopic.)

Hint. First argue we can approximate f by a smooth map $g : X \to \mathbb{R}^{n+1}$. Then normalize g, proving first that $0 \notin g(X)$.

7. Let M be a smooth manifold, $f : M \to \mathbb{R}^s$ be a C^1 map, $N \subset \mathbb{R}^s$ a submanifold of codimension strictly greater than $dim(M)$. Show that for almost every $v \in \mathbb{R}^s$ the translated image $f(M) + v$ has empty intersection with N. (That is, the set of $v \in \mathbb{R}^s$ for which the intersection is *not* empty has measure zero in $\mathbb{R}^s$.)

8. Show that if $h : S^n \to S^n$ is homotopic to a constant, then h has a fixed point and h maps some point x to its antipode $-x$. (You may assume h is smooth.) If the fact that two maps are homotopic is used in your proof, include the homotopy between them.

9. (i) Define 'homotopy equivalence' and 'deformation retraction', and prove that if X deformation retracts to a subspace $A \subset X$, then X and A are homotopy equivalent. (Note $r = i_A \circ r$ if $r : X \to X$, $r(X) = A$, is a retraction; where $i_A : A \to X$ is the inclusion map.)

(ii) Suppose there exists a deformation retraction from the space X to a point $x_0 \in X$. Show that for each open neighborhood U of x_0, there exists a second open neighborhood $V \subset U$ of x_0, with the property that the inclusion:

$$i_* : \pi_1(V, x_0) \to \pi_1(U, x_0)$$

is trivial.

10. Let M be a compact orientable surface of genus 2. Prove there exists $f : M \to S^1$ continuous, which does not lift to a continuous map from M to $\mathbb{R}$. (Here 'lift' refers to the exponential covering map $\mathbb{R} \to S^1$.)

You may use diagrams to explain the steps in your proof.

Topology Preliminary Examination

August 2020

You may omit two questions from each part

Include justifications with your answers

Part A

1. Let X be a regular topological space, and let x, y be distinct points of X. Prove that x, y have neighborhoods whose closures are disjoint.

2. Let X, Y be topological spaces, with Y compact.

(a) (6 pts) Prove that the projection $\pi_1 : X \times Y \to X$, $(x, y) \mapsto x$ is a closed map.

(b) (4 pts) Let $f : X \to Y$ be a function, not assumed to be continuous. The *graph of* f is the following subset of $X \times Y$: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. Prove that if Γ_f is closed in $X \times Y$, then f is continuous.

3. Let $\mathbb{R}$ be given the standard topology, and let $A \subset \mathbb{R}$ be the subspace $A = \bigcup_{i=1}^{\infty} \left(\frac{1}{2i+1}, \frac{1}{2i}\right]$.

(a) (6 pts) Prove that A is locally compact, and that $\mathbb{R} \setminus A$ is not locally compact.

(b) (4 pts) Find a locally compact subspace B of $\mathbb{R}$ such that $A \cup B$ is not locally compact.

4.

(a) (4 pts) Let X be a connected space such that each point of X has a path-connected neighborhood. Prove that X is path-connected.

(b) (6 pts) Give an example of a connected space that is not path-connected.

5. Let (X, d) be a metric space, and let $A \subseteq X$ be non-empty. Recall that given $x \in X$, the *distance from x to A* is $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. For the remainder of this question it is assumed that the non-empty subset A is compact.

(a) (4 pts) Prove that $d(x, A) = d(x, a)$ for some $a \in A$.

(b) (6 pts) Given $\epsilon > 0$, define $U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}$. Prove that $U(A, \epsilon)$ is an open set containing A, and that if V is any open set containing A, then V contains $U(A, \epsilon)$ for some $\epsilon > 0$.

6. Suppose that X is connected and Hausdorff, and that every proper closed subset of X containing at least two points is disconnected. Prove that $X \setminus \{x\}$ is connected for every $x \in X$.

7. Let $p : X \to Y$ be a closed, continuous, surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. Show that if X is Hausdorff, then so is Y.

8. Let $(C\,,\,d)$ be a compact metric space, and let $f : C \to C$ be a continuous function with no fixed point, *i.e.* $f(x) \neq x$ for each $x \in C$. Prove that there exists $\delta > 0$ such that $d(x\,,\,f(x)) \geq \delta$ for each $x \in C$.

Part B

9. Let X be the quotient space $B^2/\sim$, where B^2 is the unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\sim$ is the equivalence relation on B^2 generated by

$$z \sim z\,e^{2\pi i/3} \quad (|z| = 1)\ ,$$

i.e. each point z of the boundary of B^2 is identified with $\rho(z)$, where ρ is rotation through $2\pi/3$ about the center of B^2 .

(a) (5 pts) Compute the fundamental group of X .

(b) (5 pts) Prove that every continuous map from X to the projective plane P^2 is nullhomotopic.

10.

(a) (4 pts) Prove that each covering map is an open map.

(b) (4 pts) Prove that each finite-sheeted covering map is a closed map.

(c) (2 pts) Give an example, with justification, of a covering map that is not a closed map.

11. Construct 4–sheeted covering maps $p_i : E_i \to S^1 \vee P^2$ $(i = 1, 2)$, with p_1 regular, p_2 not regular and each E_i connected. Explain why your maps are covering maps and why they have the required properties.

12. Let X be a compact metric space, and let $p : \widetilde{X} \to X$ be a covering map. Prove that for some $\epsilon > 0$ every ball $B(x\,,\,\epsilon)$ in X is evenly covered.

13. Let $X\,,\,Y$ be topological spaces with respective basepoints $x_0\,,\,y_0$.
Prove that $\pi_1(X \times Y\,,\,(x_0\,,\,y_0))$ is isomorphic to $\pi_1(X\,,\,x_0) \times \pi_1(Y\,,\,y_0)$.

January 7, 2019

University of Tennessee
Topology Preliminary Examination

You may omit two questions from each part.

Part A

Question 1. Let X be an infinite set equipped with the finite complement topology. Prove that every continuous map $f : X \to \mathbb{R}$ is constant.

Question 2. Prove that the subspaces

$$X = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1]) \quad \text{and} \quad Y := ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$$

of $\mathbb{R}^2$ are not homeomorphic.

Question 3. Let A and B be disjoint compact subspaces of a Hausdorff space X. Prove that there exist disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$.

Question 4. Prove that every separable metric space is second countable. Deduce that the Sorgenfrey line $\mathbb{R}_\ell$ is not metrizable.

Question 5. Let D be any countable subset of $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus D$ is path connected.

Question 6. Prove that $\mathbb{Q} \subset \mathbb{R}$, equipped with the subspace topology induced from the standard topology on $\mathbb{R}$, is not locally compact.

Question 7. Prove that the one-point compactification of $\mathbb{N} \subset \mathbb{R}$ (with the subspace topology) is homeomoephic to $\{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ (with the subspace topology).

Question 8. Given $a \in \mathbb{R} \setminus \{0\}$, define $\mathbb{R}/a\mathbb{Z}$ as the quotient of $\mathbb{R}$ by the equivalence relation

$$x \sim y \iff y = x + ka \quad \text{for some} \quad k \in \mathbb{Z}\,.$$

Show that $\mathbb{R}/a\mathbb{Z}$ is homeomorphic to S^1 .

Part B

Question 9. Let A be a path connected subspace of $\mathbb{R}^n$, let Y be a path connected topological space and let $h : A \to Y$ be a continuous map. Show that if h extends to a continuous map $\hat{h} : \mathbb{R}^n \to Y$, then the induced map $h_* : \pi_1(A) \to \pi_1(Y)$ is trivial.

Question 10. Describe the three double coverings of $\mathbb{R}P^2 \vee S^1$ (not necessarily by an explicit formula; a diagram may suffice) and determine which (if any) is regular.

Question 11. Prove that there are no covering maps from S^2 to $S^1 \times S^1$ or from $S^1 \times S^1$ to S^2.

Question 12. Let X be the union $S^2 \cup L$ of the standard sphere $S^2 \doteqdot \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with the vertical segment $L \doteqdot \{(0, 0, z) : |z| \leq 1\}$. Determine $\pi_1(X, b)$, where $b := (0, 0, 1)$.

Question 13. Determine the fundamental group of the Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$. Justify your answer.

August 13, 2018

University of Tennessee
Topology Preliminary Examination

You may omit two questions from each part.

Part A

Question 1. Denote by $\mathbb{R}_f$ the set of real numbers equipped with the finite complement topology. Prove the following statements.

(a) $\mathbb{R}_f$ is not Hausdorff.
(b) Every subset of $\mathbb{R}_f$ is compact.
(c) Every continuous map $f : \mathbb{R}_f \to \mathbb{R}$ is constant.

Question 2. Let X and Y be connected spaces and let A and B be proper subsets of X and Y, respectively. Show that $(X \times Y) \setminus (A \times B)$ is connected.

Question 3. Prove that $\mathbb{R}^{\mathbb{N}}$, equipped with the product topology, is not locally compact.

Question 4. Prove that a connected, locally path connected space is path connected.

Question 5. (a) Prove that every separable metric space is second countable.
(b) Prove that the Sorgenfrey line, $\mathbb{R}_\ell$, is not second countable.
(c) Deduce that $\mathbb{R}_\ell$ is not metrizable.

Question 6. Prove that the quotient of the circle $S^1 \subset \mathbb{C}$ obtained by identifying 1 with -1 is homeomorphic to the figure-8 space,

$$X := \{e^{i\theta} - i : \theta \in [0, 2\pi]\} \cup \{e^{i\theta} + i : \theta \in [0, 2\pi]\} \subset \mathbb{C}.$$

Question 7. Let (Y, d) be a complete metric space, and let X be a set. Recall that the uniform metric $\overline{d}$ on Y^X associated to d is defined by

$$\overline{d}(f, g) := \sup_{x \in X} \min\{d(f(x), g(x)), 1\}.$$

(a) Show that $(Y^X, \overline{d})$ is complete.

(b) Let X be a topological space and let (Y, d) be a complete metric space. Show that the set $C(X, Y)$ of continuous functions from X to Y, equipped with the associated uniform metric $\overline{d}$, is complete.

Question 8. Prove that $\mathbb{R}^n$ is paracompact.

Part B

Question 9. Let $X \subset \mathbb{R}^3$ be the union of k distinct lines through the origin. Prove that $\pi_1(\mathbb{R}^3 \setminus X)$ is a free group on $2k - 1$ generators.

Question 10. (a) Find a group G of homeomorphisms of S^2 which acts properly discontinuously and such that $\mathbb{R}P^2 \cong S^2/G$.
(b) Using part (a), determine $\pi_1(\mathbb{R}P^2, \cdot)$.

Justify your answers.

Question 11. (a) Describe a covering map $p : E \to B$, where E consists of a round sphere in $\mathbb{R}^3$ tangent to two distinct, disjoint round circles and B is the wedge product of $\mathbb{R}P^2$ with S^1 (not necessarily by an explicit formula; a diagram may be sufficient).
(b) Assign a suitable basepoint $e \in E$ and determine the subgroup $p_{*e}\pi_1(E, e)$ of $\pi_1(B, p(e))$.
(c) Decide whether or not the covering is regular.

Justify your answers for parts (b) and (c).

Question 12. Prove that every continuous map from $\mathbb{R}P^2$ to T^2 is nullhomotopic.

Question 13. Prove *one* of the following two statements:
(A) There is no retraction from the closed two-ball $\overline{B}^2$ to its boundary, S^1.
(B) There is no smooth retraction from a smooth n-manifold-with-boundary, M, to its boundary, ∂M.
Deduce that every smooth map $f : \overline{B}^2 \to \overline{B}^2$ has a fixed point.

Topology Preliminary Examination Friday January 5, 2018

You may omit two questions from each part

Part A

1. Characterize those topological spaces X having the property that the diagonal

$$\Delta = \{x \times x \mid x \in X\}$$

is an open subset of $X \times X$.

2. Prove that the real line with the standard topology is connected.

3. Let $\mathcal{T}, \mathcal{T}'$ be two topologies on a set X.

(i) Suppose that $\mathcal{T} \subset \mathcal{T}'$. What does compactness of X under one of these topologies imply about compactness under the other?

(ii) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}, \mathcal{T}'$ are equal or they are not comparable.

4. Let X be a metric space.

(a) Suppose that for some $\epsilon > 0$ every ϵ-ball in X has compact closure. Show that X is complete.

(b) Suppose that for each $x \in X$ there is an $\epsilon > 0$ such that the ball $B(x, \epsilon)$ has compact closure. Show by means of an example that X need not be complete.

5. Let $p : X \to Y$ be a closed continuous surjective map, such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. Show that if Y is compact, then so is X.

6. Let $\mathbb{R}_\ell$ be the real line with the lower limit topology. Prove that $\mathbb{R}_\ell$ is not metrizable. (*Hint:* Consider whether $\mathbb{R}_\ell$ has the properties "separability", "second countability".)

7. Prove or disprove each of the following.

(i) If X is a complete metric space that is also bounded, then X is compact.

(ii) If A is a closed subspace of a limit point compact space X, then A is limit point compact.

(iii) A dense subspace of a complete metric space is complete.

8. Determine whether $\mathbb{R}^\omega$ is connected in the uniform topology.

Part B

9. Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$.

10. Let $B = P^2 \vee P^2$, *i.e.* B is the result of gluing together two copies of the projective plane P^2 at a single point.

(i) Find a connected three-sheeted covering space of B that is not regular, and use it to show that the fundamental group of B is not commutative.

(ii) Find a connected, regular, infinite sheeted covering space of B, and use it to show that the fundamental group of B is infinite.

11. Compute the fundamental group of the Klein bottle, either (i) using the Seifert-van Kampen theorem, or (ii) considering the Klein bottle as the quotient of the Euclidean plane under a suitable group action.

12. Show that if $h : S^1 \to S^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

13. Let $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is a simply connected open subspace of X, and where $X_n \subset X_{n+1}$ for each $n \geq 1$. Show that X is simply connected.

Topology Preliminary Examination Friday August 18, 2017

You may omit two questions from each part

Part A

1. Let X be a metric space, and let f be a function from X to a topological space Y with the following property: whenever (x_n) is a sequence in X converging to $x \in X$, the sequence $(f(x_n))$ converges to $f(x)$. Prove that f is continuous.

2. Define the terms *connected space, path connected space.*

Let $X = [0,1] \times [0,1]$, with the lexicographic (dictionary) order topology.

(i) Prove that X is connected (you may assume that X has the least upper bound property). *Hint:* Adapt the proof that $\mathbb{R}$ is connected.

(ii) Prove that X is not path connected.

3. Let X be a countable product of copies of $[0,1]$ with the uniform topology. Prove that X is not locally compact.

4. Prove or disprove each of the following.

(a) Let $f : X \to Y$ be a map of topological spaces that is continuous and surjective. If X is locally compact, then so is Y.

(b) Let $f : X \to Y$ be a map of topological spaces that is continuous, open and surjective. If X is locally compact, then so is Y.

5. Let X, Y be spaces with X locally compact Hausdorff, and let $\mathcal{C}(X,Y)$ have the compact-open topology. Show that the evaluation map $e : X \times \mathcal{C}(X,Y) \to Y$, $e(x,f) = f(x)$ is continuous.

6. For each of the following, either provide an example with justification, or prove that no example exists.

(a) A metric space X that is not second countable.

(b) A metric space X that is not first countable.

(c) A metric space X that is not normal.

7. Let X be the Hilbert cube I^ω, a countable product of copies of $I = [0, 1]$ endowed with the product topology. Assuming that each I^n $(n = 1, 2, \ldots)$ has the fixed point property, prove that X has the fixed point property. (A space X is said to have the fixed point property if each continuous function $f : X \to X$ has a fixed point.)

8. Let A, B be proper subsets of X, Y respectively. If X and Y are connected, show that $(X \times Y) \setminus (A \times B)$ is connected.

Part B

9. Let B be a compact space, and let $p : E \to B$ be a covering map. Show that if $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

10. Let X be the union of the three coordinate axes in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

11. Let $X = P^2 \vee S^1$, with basepoint x_0 taken to be the point where P^2 and S^1 are glued together.

(i) Find two inequivalent 3-fold connected covering spaces of X, illustrating each by means of a rough sketch.

(ii) Let $p_i : \widetilde{X}_i \to X$ $(i = 1, 2)$ be the covering maps of part (i) of this question. Determine whether each covering space $\widetilde{X}_i$ is regular, and specify the image of each p_{i*} in $\pi_1(X, x_0)$.

12. Compute the fundamental group of the Klein bottle, either (i) using the van Kampen theorem, or (ii) considering the Klein bottle as the quotient of the Euclidean plane under a suitable group action.

13. Let n be a prime number. Let X be a path connected space whose fundamental group is cyclic of order n, and suppose that $p : \widetilde{X} \to X$ is an $(n+1)$-sheeted covering map. Show that $\widetilde{X}$ is not path connected, and that some path component of $\widetilde{X}$ is homeomorphic to X.

Topology Preliminary Examination Monday January 9, 2017

You may omit two questions from each part

Part A

1. Let X, X' be topological spaces whose underlying sets are equal, and whose topologies are Ω, Ω' respectively. Assume that $\Omega' \supset \Omega$ and that both spaces are T_1. Consider the properties "Hausdorff", "regular", "normal". For each of these properties, if one of the spaces has the property, what does that imply about the other space?

2. Let X be a countably infinite space with the discrete topology. Show that the one-point compactification of X is homeomorphic with the subspace $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

3. Let A, B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U, V containing A, B respectively. *Hint:* First consider the case where B is a point.

4.

(a) Show that every metrizable space with a countable dense subset has a countable basis.

(b) Use part (a) of this question to show that the real line with the lower limit topology is not metrizable.

5.

(a) Prove the *Intermediate Value Theorem*: Let X be a connected space, let Y be an ordered set with the order topology, and let $f : X \to Y$ be a continuous function. If a, b are points of X and $r \in Y$ satisfies $f(a) < r < f(b)$, then there exists $c \in X$ with $f(c) = r$.

(b) Let $Y = [0,1] \times [0,1]$ be given the dictionary (lexicographic) order topology. Show that Y is not path connected.

6. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. (Such a map is called a perfect map.) Show that if X is Hausdorff, then so is Y.

7. For each of the following, either provide an example with justification, or prove that no example exists.

(a) A metric space X that is not second countable.

(b) A metric space X that is not first countable.

(c) A metric space X that is not normal.

8.

(a) Define the terms "Cauchy sequence in a metric space", "complete metric space".

(b) Show that for a metric space X, if each Cauchy sequence in X has a convergent subsequence, then X is complete.

(c) Show that for a metric space X, if there exists $\epsilon > 0$ such that each ϵ-ball in X has compact closure, then X is complete.

Part B

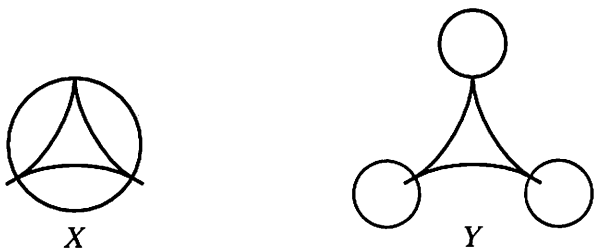
9.

(a) Prove that each covering map is an open map.

(b) Prove that each finite-sheeted covering map is a closed map.

(c) Give an example, with justification, of a covering map that is not a closed map.

10.



Illustrated are subsets X, Y of the plane; X consists of a circle with an inscribed triangle, and Y consists of a triangle with a circle touching each of its vertices.

(a) Describe covering maps $p : X \to S^1 \vee S^1$, $q : Y \to S^1 \vee S^1$. (You do not need to provide explicit formulae for p, q, but describe how the various arcs constituting X, Y are mapped into $S^1 \vee S^1$.)

(b) Define the term *regular covering map*, and determine whether the covering maps p, q of part (a) are regular.

(c) Assigning suitable basepoints to the spaces concerned, determine the subgroups $p_*(\pi_1(X))$, $q_*(\pi_1(Y))$ of $\pi_1(S^1 \vee S^1)$, where p, q are the covering maps of part (a).

11. Let X, Y be spaces with respective basepoints x_0, y_0. Prove that $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

12. Suppose that C is a compact subset of the real line $\mathbb{R}$ and that $f : C \to S^1$ is continuous. Prove that there exists a continuous function $g : C \to \mathbb{R}$ such that $f(c) = (\cos(g(c)), \sin(g(c)))$ for all $c \in C$.

13.

(a) Determine the fundamental group of the projective plane P. Here you may assume that S^2 is simply connected and that covering maps have the path homotopy lifting property, but you may not assume the Seifert-van Kampen theorem.

(b) Let $p : E \to B$ be a covering map, and let $f : Y \to B$ be a continuous map. Carefully state a theorem giving sufficient conditions on f for existence of a lifting of f to E.

(c) Use part (a) together with the theorem of part (b) to show that every continuous map $p : P \to S^1$ is nullhomotopic.

Topology Preliminary Examination Wednesday August 10, 2016

You may omit two questions from each part

Part A

1. Show that if X is regular, then any two distinct points of X have neighborhoods whose closures are disjoint.

2. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. (Such a map is called a perfect map.) Show that if X is second-countable, then so is Y.

3. Let $\mathbb{R}_\ell$, $\mathbb{R}$ be the real line endowed with the lower limit topology and the standard topology, respectively. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

4. Recall that a topological space X is *locally compact* if for each point $x \in X$ there exists a compact subspace of X containing a neighborhood of x. If X is locally compact and $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open? Justify your answer.

5. An *isometry* of a metric space (X, d) is a map $f : X \to X$ satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

(a) Prove that if (X, d) is compact and $f : X \to X$ is an isometry, then f is a homeomorphism.

(b) Give an example of a metric space (X, d) and an isometry of X that is not a homeomorphism.

6. Let $\mathcal{T}$, $\mathcal{T}'$ be two Hausdorff topologies on a set X.

(a) Suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) Show that if X is compact under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$, $\mathcal{T}'$ are equal or they are not comparable.

7. Let C be the standard middle-thirds Cantor set in $[0, 1]$, and let $f : C \to [0, 1]$ be any continuous function. Show that there exists a continuous function $g : C \to [0, 1]$ such that $f(c) \neq g(c)$ for all $c \in C$.

8. A space Z *has the fixed point property* if for every continuous function $g : Z \to Z$ there exists $z \in Z$ such that $g(z) = z$.

(a) Prove that a retract of a space with the fixed point property has the fixed point property.

(b) Suppose that X and Y are spaces with the fixed point property. If each of X, Y is closed in $X \cup Y$ and $X \cap Y = \{p\}$ is a point, show that $X \cup Y$ has the fixed point property.

Part B

9. Suppose that C is a compact subset of the real line $\mathbb{R}$ and that $f : C \to S^1$ is continuous. Prove that there exists a continuous function $g : C \to \mathbb{R}$ such that $f(c) = (\cos(g(c)), \sin(g(c)))$ for all $c \in C$.

10. Suppose that a Hausdorff space X is the union of two simply connected, path connected open subsets U, V such that the intersection $U \cap V$ is non-empty and path connected. Prove that $\pi_1(X, x_0)$ is the trivial group (where you may assume that $x_0 \in U \cap V$). If you use the Seifert-van Kampen theorem for this, prove it.

11. Describe the three 2-sheeted connected covers of $P^2 \vee S^1$, where P^2 denotes the projective plane. Determine which of these covers (if any) is a regular cover, *i.e.* corresponds to a normal subgroup of $\pi_1(P^2 \vee S^1)$.

12. Calculate the fundamental group of $\mathbb{R}^3 \setminus L$, where L is a finite union of lines through the origin.

13. Let n be a prime number. Let X be a path connected space whose fundamental group is cyclic of order n, and suppose that $p : \widetilde{X} \to X$ is an $(n+1)$-sheeted covering map. Show that $\widetilde{X}$ is not path connected, and that some path component of $\widetilde{X}$ is homeomorphic to X.

University of Tennessee
Topology Preliminary Examination
January 8, 2016

You must omit two questions from each part. All of your work should appear on the separate paper provided. For each answer you provide, clearly indicate which problem you are answering. Fully justify all of your answers.

Part I

Problem 1. *Let $f : X \to Y$ be a closed, surjective, continuous map such that $f^{-1}(\{y\})$ is compact for each $y \in Y$.*

(i). *Prove that if X is Hausdorff, then so is Y.*

(ii). *Prove that if Y is compact, then so is X.*

Problem 2. *Let $X \subset \mathbb{R}^2$ be the union of the closed line segment joining (0,0) and (1,0) together with the closed line segments joining $(0,0)$ and $(1, \frac{1}{n})$, $n = 1, 2, 3, \ldots$. Show that for any continuous map $r : \mathbb{R}^2 \to X$ there is a point $z \in X$, with $r(z) \neq z$.*

Problem 3. *Let D be any countable subset of $\mathbb{R}^2$, the plane with the usual topology. Prove that $R^2 \backslash D$ is path connected.*

Problem 4. *A subset C of the plane is called* convex *if for any $x, y \in C$, the line segment joining x and y is contained in C.*

(i – 3 points). *Give a topological classification of all compact convex subsets of $\mathbb{R}$. If you use a theorem, prove it.*

(ii – 7 points). *Give a topological classification of all compact convex subsets of the plane. If you use a theorem, prove it.*

Problem 5. *Suppose $X \subset \mathbb{R}^3$ and X is not homeomorphic to a proper subset of itself. Prove X has empty interior.*

Problem 6. *Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\vec{x}$ and $\vec{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence*

$$\vec{x} - \vec{y} = (x_1 - y_1, x_2 - y_2, \ldots)$$

is bounded.

Problem 7. *Consider the space $C(\mathbb{R}, \mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the compact-open topology. show that the map $\mathcal{F} : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$,*

$$\mathcal{F}(f) = f(0),$$

is continuous.

Problem 8. *Suppose that* $\{X_\alpha,\ \alpha \in J\}$, *is a collection of connected spaces. Write*

$$X := \prod_{\alpha \in J} X_\alpha$$

and give X *the product topology. Show that* X *is conneted.*

Part II

Problem 9.

(i). *Let* X, Y *be spaces with respective basepoints* x_0, y_0. *Prove that*

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

(ii). *Let* P *denote the projective plane. Determine, up to equivalence, all connected covering spaces of* $P \times P$.

Problem 10. *Given a loop* $\alpha : [0,1] \to S^1$ *at* $(1,0)$, *its* degree $\deg(\alpha)$ *is the integer* n *such that the lift of* α *to the reals starting at* 0 *ends at* n. *This is under the covering map* $p : \mathbb{R} \to S^1$ *defined by* $p(t) = (\cos(2\pi t), \sin(2\pi t)) = e^{2\pi t \cdot i}$. *Show that* $h : \pi_1(S^1, 1) \to Z$ *given by* $h([\alpha]) = \deg(\alpha)$ *is an isomorphism of groups.*

Problem 11. *Consider the sphere* $S^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ *and write* Y *for that portion of the* z*-axis determined by* $-1 \leq z \leq 1$. *Find the fundamental group of the space* $S^2 \cup Y$ *at the point* $(0,0,1)$.

Problem 12. *Show that any continuous map from* S^2 *to the torus* $S^1 \times S^1$ *is homotopic to a constant map.*

Problem 13.

(i). *Show that there are no covering maps* $S^2 \to S^1 \times S^1$.

(ii). *Show that there are no covering maps* $S^1 \times S^1 \to S^2$.

University of Tennessee
Topology Preliminary Examination
August 12, 2015

You may omit two questions from each part. Remember to fully justify, with proof, all of your answers.

Part I

Question 1. *Let X be a space and consider $x_0 \in X$. Consider the inclusion $i : X \to X \times X$, given by*

$$i(x) = (x, x).$$

Find a necessary and sufficient condition on X for i to be a closed map.

Question 2. *Let X be a space.*

(a). *Show that each connected component of X is closed.*

(b). *Show that if the number of connected components is finite, then each connected component is open.*

(c). *Give an example of a space X and a connected component $C \subset X$ with the property that C is not open.*

Question 3. *Suppose that (X, d) is a metric space, and suppose $x_0 \in X$. Define $B := \{x : d(x, x_0) < 1\}$ and define $C := \{x : d(x, x_0) \leq 1\}$.*

(a). *Is C compact?*

(b). *Prove or show false: C is the closure of B.*

Question 4. *Suppose that X is a space and $A \subset X$ is a connected subset. Prove or show false: There exist subsets U and V, open in X, such that $U \cap V = \emptyset$ and $A \subset U \cup V$.*

Question 5. *Suppose that X is locally compact and Hausdorff. Prove that for every $x_0 \in X$ and every open subset $U \ni x_0$, there exists an open set $V \ni x_0$ with the property that $\overline{V}$ is compact and $\overline{V} \subset U$. Hint: Consider the one-point compactification of X.*

Question 6. *Let X be the collection of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.*

(a). *Write $\mathcal{T}_c$ for the compact-open topology on X. Write down a subbasis for $\mathcal{T}_c$ and draw a picture of a typical subbasis element. That is to say: explicitly choose an element of the subbasis you wrote down, then draw the graphs of several functions in this element on the same set of axes.*

(b). *Since each $f \in X$ may be regarded as an element of $\mathbb{R}^{\mathbb{R}}$, we may view X as a subspace of $\mathbb{R}^{\mathbb{R}}$ with the product topology, which we denote $\mathcal{T}_p$. Once again, write down a subbasis for this topology and draw a picture of a typical subbasis element.*

(c). *Is $\mathcal{T}_c = \mathcal{T}_p$? Are they comparable? Justify your answers with a proof.*

Question 7. *Show that $\mathbb{R}^{\mathbb{R}}$ with the product topology is not metrizable.*

Question 8. *Give an example of each of the following:*

(a). *Two non-homeomorphic spaces, neither of which is Hausdorff*

(b). *Two spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.*

(c). *A totally disconnected space X with the property that the topology on X is not the discrete topology.*

Part II

Question 9. *Explicitly (i.e. write a formula) find the universal cover of the Möbius band. Justify your answer with a proof.*

Question 10.

(a). *Write $X := [0,1] \times [-1,1]$ and consider the equivalence relation determined by declaring the pair $(0,t)$ equivalent to $(1,t)$, $-1 \leq t \leq 1$, and declaring every other point of X equivalent to itself. Write $Y := X/\sim$ for the quotient space. Show that Y is homeomorphic to $S^1 \times [-1,1]$.*

(b). *Show that the cylindrical segment $S^1 \times [-1,1]$ is a double cover of the Möbius strip.*

Question 11. *Suppose that there is a deformation retraction from the space X to a point $x_0 \in X$. Show that for each open set $U \ni x_0$, there is an open set $V \ni x_0$, $V \subset U$, with the property that the inclusion*

$$i_* : \pi_1(V, x_0) \to \pi_1(U, x_0)$$

is trivial.

Question 12. *Write $X := \mathbb{R}^2 \backslash (0,0)$. Choosing any basepoint that you like, carefully calculate the fundamental group of $X \vee S^1$.*

Question 13. *Make mathematically precise, and then prove, the following statement. 'In a path-connected topological space, it doesn't matter what basepoint you pick to calculate the fundamental group.'*

Topology Preliminary Examination Monday January 5, 2015

You may omit two questions from each part

Part A

1. Let X be a topological space with the property that for any two distinct points of X, there is an open set containing exactly one of them. Suppose also that for any $x \in X$ and any closed subset A of X not containing x, there are disjoint open sets $U \ni x$ and $V \supseteq A$. Prove that X is Hausdorff.

2. Let $\overline{d}(a, b) = \min(|a - b|, 1)$ be the standard bounded metric on $\mathbb{R}$. Recall that the *uniform metric* $\overline{\rho}$ on $\mathbb{R}^\omega$ is defined as follows: given points $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$ in $\mathbb{R}^\omega$, $\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup \overline{d}(x_n, y_n)$.
Given a point $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and a number $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = \prod_{n \in \mathbb{N}} (x_n - \epsilon, x_n + \epsilon).$$

(i) Show that $U(\mathbf{x}, \epsilon)$ is not open in the metric space $(\mathbb{R}^\omega, \overline{\rho})$.

(ii) Give a description of the open ball of radius δ about $\mathbf{x}$ in $(\mathbb{R}^\omega, \overline{\rho})$, in terms of sets of form $U(\mathbf{x}, \epsilon)$.

3. Let C be the standard middle-thirds Cantor set in $[0, 1]$, and let $f : C \to [0, 1]$ be any continuous function. Show that there exists a continuous function $g : C \to [0, 1]$ such that $f(c) \neq g(c)$ for all $c \in C$.

4. State the *Lebesgue Number Lemma.* Using this result or otherwise, prove the following: Let X, Y be metric spaces, and suppose that X is compact. Then each continuous function $f : X \to Y$ is uniformly continuous.

5. Let X be the Hilbert cube I^ω, a countable product of copies of $I = [0, 1]$ endowed with the product topology. Assuming that each I^n $(n = 1, 2, \ldots)$ has the fixed point property, prove that X has the fixed point property. (A space X is said to have the fixed point property if each continuous function $f : X \to X$ has a fixed point.)

6. Give an example, with justification, of a Hausdorff space that is not metrizable.

7. Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction mapping, *i.e.* there exists $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Prove that f has a unique fixed point.

8. Let $\mathbb{R}_\ell$ be the real line with the lower limit topology, *i.e.* the topology with basis the collection of all half-open intervals $[a, b)$ $(a, b \in \mathbb{R}, a < b)$.

(i) Show that the space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable, *i.e.* it has a countable dense subset.

(ii) Find a closed subset of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ that is not separable.

Part B

9(i) Carefully define the *fundamental group* of a space X with basepoint $x_0 \in X$, and explain how a continuous function $f : X \to Y$ gives rise to a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

(ii) Show that if $f : S^1 \to S^1$ is nullhomotopic, then f has a fixed point and f maps some point x to its antipode $-x$.

10. Compute the fundamental group of the Klein bottle, either (i) using the van Kampen theorem, or (ii) considering the Klein bottle as the quotient of the Euclidean plane under a suitable group action.

11. Let $p : E \to B$ be a covering map.

(i) Show that p is an open map, and give an example to show that p is not necessarily a closed map.

(ii) Show that if B is Hausdorff, then so is E.

12. Let X be a path connected, locally path connected space whose fundamental group is finite. Prove that any map $f : X \to S^1$ is nullhomotopic. (State carefully any "big" theorem that you use.)

13. Let P denote the projective plane, and let $X = P \vee P$, two copies of P glued together at a single point x_0. By lifting paths to a suitable covering space of X, show that $\pi_1(X, x_0)$ is not abelian. Also determine whether or not your covering space is regular.

Topology Preliminary Examination Wednesday August 13, 2014

You may omit two questions from each part

Part A

1. Let X be an infinite set with the finite complement topology. Prove that every continuous map $f : X \to \mathbb{R}$ is constant.

2. Let X be a topological space. Prove that the following are equivalent:

(a) X is T_1, *i.e.* every point in X is closed.

(b) Every subset A of X is the intersection of a family of open sets in X.

3. Assuming that each cell I^n has the fixed point property, prove that the Hilbert cube I^∞ has it as well.

4. Let X, Y be the following subspaces of $\mathbb{R}^2$:

$$X = (\{0\} \times [-1,1]) \cup ([-1,1] \times \{0\}) \quad , \quad Y = (\{0\} \times [0,1]) \cup ([-1,1] \times \{0\}) \ .$$

Show that X, Y are not homeomorphic.

5. Let X be a connected space, and let $f, g : X \to [0,1]$ be continuous functions with f surjective. Prove that there exists $x \in X$ with $f(x) = g(x)$.

6.

(a) Give an example, with justification, of a continuous function $f : X \to Y$ such that Y is path-connected and $f^{-1}(y)$ is path-connected for all $y \in Y$, but X is not path-connected.

(b) Let $g : A \to B$ be continuous, where B is path-connected and $g^{-1}(b)$ is path-connected for all $b \in B$. Suppose that there exists a continuous function $h : B \to A$ such that $g \circ h$ is the identity on B. Show that A is path-connected.

7. Is it true that if the 1-point compactifications of two locally compact Hausdorff spaces X, Y are homeomorphic, then X, Y are necessarily homeomorphic? Give a proof or counterexample, as appropriate.

8. Let X be a complete metric space.

(a) Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ be a nested sequence of non-empty, closed, bounded subsets of X, whose diameters $\to 0$. Prove that $\bigcap_{m=1}^{\infty} F_m$ consists of a single point.

(b) Prove that X is a *Baire space, i.e.* the intersection of a countable family of open dense subsets of X is dense in X.

Part B

9. Let X be the union of the three coordinate axes in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

10. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let Y be a path-connected, locally path-connected space, and let $f : Y \to B$ be a continuous function such that $f(y_0) = b_0$ and $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$. Prove that f can be lifted to a continuous function $F : Y \to E$ such that $F(y_0) = e_0$ and $p \circ F = f$.

11. Let G be a topological group, with group operation $\cdot$ and identity element e. Given loops f, g in G based at e, let $f \cdot g$ denote their pointwise product $(f \cdot g)(t) = f(t) \cdot g(t)$. Show that this operation on loops based at e induces a well-defined operation on path homotopy classes, and that this induced operation is the same as the group operation of $\pi_1(G, e)$. Show that $\pi_1(G, e)$ is Abelian.

12.

(a) Compute the fundamental group of $X = P^2 \vee S^1$, the one-point union of a projective plane and a circle.

(b) Either prove that X has precisely three distinct connected 2-fold covering spaces, or construct two distinct connected 2-fold covering spaces of X.

13. Let S^2 be the unit sphere in $\mathbb{R}^3$, *i.e.* $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$. Consider the map $f : A \times B \to S^2$, where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid (y - 3)^2 + z^2 = 1, x = 0\}$$

and

$$f(p, q) = \frac{p - q}{\|p - q\|} \qquad (p \in A, q \in B).$$

Is the map f homotopic to a constant map?

TOPOLOGY PRELIMINARY EXAMINATION

AUGUST 16, 2013

You may omit two questions from each part (the result of the prelim is based on top 6 problems from Part I and top 3 problems from Part II).

PART I

1. For a set X, let X_f stand for X equipped with the finite complement topology. Let X, Y be two sets. Compare the topologies of $(X \times Y)_f$ and $X_f \times Y_f$. Are they equal? Is one of them strictly finer than the other?

2. Suppose that f and g are two continuous maps from a topological space X to a Hausdorff space Y. Let $Z \subset X$ be the subspace consisting of all points $x \in X$ such that $f(x) = g(x)$. Show that Z is closed in X.

3. A Hausdorff space X is called completely normal if every subspace of X is normal. Prove that the following are equivalent:
(a) X is completely normal.
(b) For every $A; B \subset X$ such that $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$, there are open sets $U; V$ such that $A \subset U, B \subset V$, and $U \cap V = \emptyset$.

4. Show that the subspaces [0,1) and (2,3) of the real line $\mathbb{R}$ are not homeomorphic.

5. Let X be the so-called Hawaiian earring, which is the subspace of the plane defined by $X = \cup_{n=1}^{\infty} C_n$, where
$$C_n = \left\{ (x, y) \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}.$$
Define an equivalence relation on the line $\mathbb{R}$ as follows: $a \sim b$ if either $a = b$ or if $a, b \in \mathbb{Z}$. Let Y be the corresponding quotient space of $\mathbb{R}$.
Show that the spaces X and Y are not homeomorphic.

6. Which of the three subsets $P, Q, P \cap Q$ of R^2 are connected? Here
$$P = \{(x; y) | \text{ at least one of } x; y \text{ is irrational}\}$$
and
$$Q = \{(x; y) | \text{ at least one of } x; y \text{ is rational}\}$$

7. Suppose $\{f_\alpha : X \to Y_\alpha \mid \alpha \in A\}$ is a collection of continuous functions defined on a Hausdorff space X such that to each $x \in X$ and each neighborhood U of x, there corresponds $\alpha \in A$ such that $f_\alpha(x) \notin Clos(f_\alpha(X - U))$. Show that X can be embedded in $\prod_\alpha Y_\alpha$.

8. Let A_n, $n \geq 1$, be closed subsets of the line $\mathbb{R}$ such that $A_i \cap A_j = \emptyset$ for any $i \neq j$. Show that

$$Int\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} Int(A_n).$$

PART II

9. Let $X = \cup_{n=1}^{\infty} X_n$ where X_n is a simply connected open subspace of X and $X_k \subset X_{k+1}$ for each positive integer k. Show that X is simply connected.

10. Suppose that X is a path connected space, $p : \tilde{X} \to X$ is a 3-fold covering map and suppose that the fundamental group of X is isomorphic to the cyclic group of order 2. Show that $\tilde{X}$ is not connected and some component of $\tilde{X}$ is homeomorphic to X.

11. Denote by B^2 the unit disk $\{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ and by S^1 its boundary $\{x \in \mathbb{R}^2 \mid |x| = 1\}$. Suppose $f : B^2 \to \mathbb{R}^2$ is a map such that $f(S^1) \subset B^2$. Show that $f(z) = z$ for some point $z \in B^2$.

12. Show that any map of the 2-dimensional sphere S^2 to the torus $S^1 \times S^1$ is homotopic to a constant map.

13. Denote by S^2 the unit sphere in $\mathbb{R}^3$: $S^2 = \{x \in \mathbb{R}^3 \mid ||x|| = 1\}$. Let $f : S^2 \to S^2$ be a continuous map such that $||f(p) - p|| < 1$ for all $p \in S^2$. Must f be onto?

University of Tennessee
Topology Preliminary Examination
January 4, 2013

You may omit two questions from each part (the result of the prelim is based on top 6 problems from Part I and top 3 problems from Part II).

PART I

- 1. Consider the following topologies on R:
  (i) discrete,
  (ii) standard (order topology), and
  (iii) Zariski (finite complement topology).
  For each of these topologies, determine whether or not the interval $(-\infty, 2012)$ is:
  (a) open,
  (b) closed,
  (c) compact (in the subspace topology).

- 2. Let $\{A_\alpha\}$ be a collection of subsets of a topological space X satisfying $X = \bigcup A_\alpha$. Suppose $f : X \to Y$ is a function to a topological space Y such that $f|A_\alpha : A_\alpha \to Y$ is continuous for each α.
  Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous. An indexed family of sets $\{A_\alpha\}$ is said to be **locally finite** if each point x of X has an open neighborhood that intersects A_α for only finitely many values of α.

- 3. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) \setminus (A \times B)$ is connected.

- 4. Prove or disprove each of the following statements:
  (i) Each metrizable space has a countable basis for its topology.
  (ii) If X, Y are homeomorphic metric spaces and X is complete, then Y is complete.
  (iii) If the metric space X is compact, then it is complete.

- 5. **(a)** Let X be a topological space, and let $\Delta \subset X \times X$ be the subset $\{(x, x) : x \in X\}$. Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

  (b) A *topological group* is a group G that is also a topological space, satisfying the following conditions: (i) G is T_1, *i.e.* singleton subsets of G are closed; (ii) the map

$\mu : G \times G \to G$ sending (g_1, g_2) to $g_1 g_2$ is continuous; (iii) the map $\tau : G \to G$ sending g to g^{-1} is continuous.

Use part (a) of this question to show that a topological group is Hausdorff.

- 6. Let $p : X \to Y$ be a quotient map (that means $U \subset Y$ is always open if $p^{-1}(U)$ is open in X). Show that if X is locally connected, then Y is locally connected.

- 7. Let $x_1, x_2, \ldots$ be a sequence of points of the product space $\prod X_\alpha$. Show that this sequence converges to the point x if and only if the sequence $\pi_\alpha(x_1)$, $\pi_\alpha(x_2)$, $\ldots$ converges to $\pi_\alpha(x)$ for each α.

- 8. Show that if X is regular, every pair of different points of X have neighborhoods whose closures are disjoint.

PART II

- 9. (i) Carefully define the *fundamental group* of a space X with basepoint $x_0 \in X$, and explain how a continuous function $f : X \to Y$ gives rise to a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

  (ii) Show that if $f : S^1 \to S^1$ is nullhomotopic, then f has a fixed point and f maps some point x to its antipode $-x$.

- 10. Compute the fundamental group of the space obtained from the unit 2-dimensional sphere by identifying the North and South Poles.

- 11. Define $S^1 \vee S^1$ (called the wedge of two copies of S^1) as $S^1 \times 1 \cup 1 \times S^1 \subset S^1 \times S^1$. Show that $\pi_1(S^1 \vee S^1)$ is not Abelian without using Seifert-van Kampen theorem.

- 12. Define contractible spaces. Show the space of reals R is contractible.

- 13. Let $p : X' \to X$ be a covering projection and let A be a connected space. Suppose $f : A \to X$ is a map such that $f(A)$ is contained in an evenly covered open subset of X. Show that any two lifts $f', f'' : A \to X'$ of f are identical if $f'(a_0) = f''(a_0)$ for some $a_0 \in A$.

University of Tennessee
Topology Preliminary Examination
August 13, 2012

You may omit two questions from each part (the result of the prelim is based on top 6 problems from Part I and top 3 problems from Part II).

PART I

- 1. Let X be a **first countable space** (that means every point in X has a countable basis of neighborhoods).
  a. Define convergence $x_n \to x_0$ in X.
  b. Show a function $f : X \to Y$ of first countable spaces is continuous if and only if $x_n \to x_0$ in X implies $f(x_n) \to f(x_0)$ in Y.

- 2. Suppose Y is a topological space consisting of exactly 2 points. Show that for any connected subset X_0 of a topological space X and any continuous function $f : X_0 \to Y$ there is a continuous extension $F : X \to Y$ of f (that means $F(x) = f(x)$ for all $x \in X_0$).

- 3. Characterize topological spaces X such that the diagonal $\Delta(X)$ is an open subset of $X \times X$.

- 4. Let (X, d) be a compact metric space with the property that for any $t < 1$, there are points x_t, y_t so that $d(x_t, y_t) = t$. Prove there are points x and y so that $d(x, y) = 1$.

- 5. Let X be locally path connected. Show that every connected open set in X is path connected.

- 6. A topological space is called a **Baire space** if the union of any countable collection of closed sets with empty interior has empty interior. Show that if Y is a G_δ set in X, and if X is compact Hausdorff, then Y is a Baire space in the subspace topology.

- 7. Prove that $[0, 1) \times [0, 1)$ is homeomorphic to $[0, 1] \times [0, 1)$.

- 8. (a) List all **separation axioms** you know.
  (b) Suppose every closed family $\{F_s\}_{s \in S}$ such that $\{int(F_s)\}_{s \in S}$ is a cover of a topological space X has a finite subcover. Under what separation axioms on X is X compact?

PART II

- 9. Suppose Y is a connected space consisting of exactly 2 points.
  a. Show Y is path connected.
  b. Show Y is simply connected.

- 10. (a) Define the property of existence and uniqueness of **path lifting** for a map $p : E \to B$ of topological spaces.
  (b) Suppose $p : E \to B$ is a map, $E \neq \emptyset$, and $B = [0, 1]$. Show p has existence and uniqueness of path lifting if and only if $p|C : C \to B$ is a homeomorphism for every path-component C of E.

- 11. Suppose $f : S^1 \to R$ is continuous. Show there is $z_0 \in S^1$ such that $f(-z_0) = f(z_0)$.

- 12. Show that every map $f: RP^2 \to S^1 \times S^1$ from the projective plane to a torus is homotopic to a constant map.

- 13. Let $p : E \to B$ be a map of topological spaces. Suppose U and V are subsets of B **evenly covered** by p. Show $U \cup V$ is evenly covered if $U \cap V$ is non-empty and connected.

Topology Preliminary Examination Friday January 6, 2012

You may omit two questions from each part

Be sure to give full explanations

Part A

1. Suppose Y is an ordered set with the order topology. Let X be a subset. Prove or give a counterexample: the order topology and the subspace topology on X coincide.

2. Let X be a space. Let us define a relation $\sim$ on X by saying that $x \sim y$ if there is no separation of X into disjoint open sets A, B such that $x \in A$ and $y \in B$. Prove that the relation $\sim$ is an equivalence relation; the equivalence classes are the *quasicomponents* of X. Find a subspace X of the plane with the following property: there exists a quasicomponent of X that is not a component of X.

3(i) Let $\mathbb{R}^\omega$ be given the product topology. Prove that if K is a compact subset of $\mathbb{R}^\omega$, then K has empty interior.

(ii) Let X be the space $[0, 1]^\omega$ with the uniform topology. Show that X is not locally compact.

4. Let (f_n) be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to a function f, prove that f is continuous.

5. Let X be a metric space.

(i) Suppose that for some $\epsilon > 0$, every ϵ-ball in X has compact closure. Show that X is complete.

(ii) Suppose that for each $x \in X$ there is an $\epsilon > 0$ such that the ball $B(x, \epsilon)$ has compact closure. Show by means of an example that X need not be complete.

6. Let $X \subset \mathbb{R}^\omega$ be the set of all sequences $\mathbf{x} = (x_i)$ such that $\sum_i x_i^2$ converges. The *ℓ^2-topology on X* is the topology induced by the metric $d(\mathbf{x}, \mathbf{y}) = \left(\sum_i (x_i - y_i)^2\right)^{1/2}$.

Show that on X, we have the inclusions box topology $\supset$ ℓ^2-topology $\supset$ uniform topology. Show that these three topologies on X are distinct.

7. Let X be a compact metric space. Prove directly that X embeds into a countable product of unit closed intervals.

8. Suppose that X is connected and Hausdorff, such that every proper closed subset of X containing at least two points is disconnected. Show that $X \setminus \{x_0\}$ is connected for every $x_0 \in X$.

Part B

9. Let $X = P^2 \vee S^1$, *i.e.* X is the result of gluing together the projective plane P^2 and the circle S^1 at a single point. Use a suitable covering space of X to show that the fundamental group of X is not commutative.

10. Let G be a topological group, with group operation $\cdot$ and identity element e. Given loops f, g in G based at e, let $f \cdot g$ denote their pointwise product $(f \cdot g)(t) = f(t) \cdot g(t)$. Show that this operation on loops based at e induces a well-defined operation on path homotopy classes, and that this induced operation is the same as the group operation of $\pi_1(G, e)$. Show that $\pi_1(G, e)$ is Abelian.

11. Use the Lebesgue number lemma to show that S^2 is simply connected. (Do not use the Seifert - van Kampen theorem.)

12. Prove (carefully) that each continuous function from the unit disk B^2 to itself has a fixed point. Further, prove that if A is a retract of B^2, then each continuous function from A to itself has a fixed point.

13. Recall that a subspace A of a topological space X is a *strong deformation retract* of X if there is a continuous map $H : X \times I \to X$ such that: (i) $H(x, 0) = x$ for each $x \in X$, (ii) $H(x, 1) \in A$ for each $x \in X$, and (iii) $H(a, t) = a$ for each $a \in A$ and each $t \in I$.

(i) Suppose that A is a strong deformation retract of X. Let $a_0 \in A$. Show that the inclusion map $j : (A, a_0) \to (X, a_0)$ induces an isomorphism of fundamental groups.

(ii) Give an example of a contractible space X and a point $x_0 \in X$, such that the subspace $\{x_0\}$ is not a strong deformation retract of X.

Topology Preliminary Examination Wednesday August 10, 2011

You may omit two questions from each part

Be sure to give full explanations

Part A

1. Define the *finite complement topology* on a set X, and prove that this is indeed a topology on X.

Let $\mathbb{R}$ be given the finite complement topology. To which point or points does the sequence $(1/n)$ converge?

2. Prove or disprove (with an explicit example) the following statement: Let A be a set with two compact, Hausdorff topologies $\mathcal{T}$ and $\mathcal{S}$. If $\mathcal{T}$ is contained in $\mathcal{S}$ then $\mathcal{T}$ equals $\mathcal{S}$.

3. Suppose that $d : X \times X \to [0, \infty)$ is a symmetric function such that $d(x, y) = 0$ if and only if $x = y$. Show that d is a metric if $d(x, z) \leq \max(d(x, y), d(y, z))$ for all $x, y, z \in X$. A space equipped with d satisfying these conditions is called an *ultrametric space.* Show that the topology for an ultrametric space has a basis consisting of sets that are both closed and open. Conclude that an ultrametric space X is totally disconnected, *i.e.* each connected subset of X contains at most one point.

4. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^\omega$, and for $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ let

$$U(\mathbf{x}, \epsilon) = \prod_{i \geq 1} (x_i - \epsilon, x_i + \epsilon) .$$

(i) Show that $U(\mathbf{x}, \epsilon)$ is not open in the uniform topology.

(ii) Let $B_{\overline{\rho}}(\mathbf{x}, \epsilon)$ denote the open ϵ-ball about $\mathbf{x}$ in the metric $\overline{\rho}$. Show that

$$B_{\overline{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta) .$$

5. Let C be the standard middle-thirds Cantor set in $[0, 1]$, and let $f : C \to [0, 1]$ be any continuous function. Show that there exists a continuous function $g : [0, 1] \to [0, 1]$ such that $f(c) \neq g(c)$ for all $c \in C$.

6. Prove that a metric space is compact if and only if it is complete and totally bounded.

7(i) Let A be a connected subset of a topological space X. Show that if $A \subset B \subset \overline{A}$, then B is connected.

(ii) Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 - A$ is path connected.

8. Given topological spaces X, Y, let $\mathcal{C}(X, Y)$ be the space of continuous functions from X to Y, with the compact-open topology.

Let Y be a locally compact Hausdorff space. Show that the "composition" map

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z) \quad , \quad (f, g) \mapsto g \circ f$$

is continuous.

Part B

9. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.

10. A continuous function $p : E \to B$ *has the unique path lifting property* if, given $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$, a path in B beginning at b_0 has a unique lift, with respect to p, to a path in E beginning at e_0.

Suppose that $p : E \to \mathbb{R}$ has has the unique path lifting property. Show that p is a homeomorphism if E is path connected. Characterize the spaces F such that the projection $F \times \mathbb{R} \to \mathbb{R}$ has the unique path lifting property.

11. Suppose that B and D are connected, locally path connected, and semilocally simply connected. Assume that A is the universal covering space of B and C is the universal covering space of D. Prove that $B \times D$ is semilocally simply connected, and that the universal covering space of $B \times D$ is $A \times C$ (all with the product topology).

12. Let X be the figure-eight space, and let Y be the theta-space, *i.e.*

$$X = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + (y+1)^2 = 1\} \quad ,$$
$$Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2 : -1 \le x \le 1\} \ .$$

Show that X, Y are homotopy equivalent.

13(i) Show that every continuous map from the projective plane P^2 to the circle S^1 is nullhomotopic.

(ii) Find a continuous map from the torus $S^1 \times S^1$ to S^1 that is not nullhomotopic.

University of Tennessee
Topology Preliminary Examination
January 7, 2011

You may omit two questions from each part (the result of the prelim is based on top 6 problems from Part I and top 3 problems from Part II).

PART I

- 1. Let R^∞ be the subset of R^ω consisting of all sequences that are 'eventually zero ', that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of R^∞ in R^ω in the box and product topologies? Justify your answer.

- 2. Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y.

- 3. **(a)** Let X be a topological space, and let $\Delta \subset X \times X$ be the subset $\{(x, x) : x \in X\}$. Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

  (b) A *topological group* is a group G that is also a topological space, satisfying the following conditions: (i) G is T_1, *i.e.* singleton subsets of G are closed; (ii) the map $\mu : G \times G \to G$ sending (g_1, g_2) to $g_1 g_2$ is continuous; (iii) the map $\tau : G \to G$ sending g to g^{-1} is continuous.

  Use part (a) of this question to show that each topological group is Hausdorff.

- 4. Show that $R \times R$ in the dictionary order topology is metrizable.

- 5. Suppose (X, ρ) is a complete, connected, locally pathwise connected metric space and $A_1, A_2, \ldots$ are compact subsets of X such that, for any path $f : [0,1] \to X$, any $\epsilon > 0$ and any integer $i > 0$ there is a path $g : [0,1] \to X$ such that $\rho(g, f) < \epsilon$ and $g([0,1]) \cap A_i = \emptyset$. Show that $X - \bigcup_i A_i$ is pathwise connected.

- 6. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

- 7. Show that $[0,1]^\omega$ is not locally compact in the uniform topology.

- 8. Show that in a Hausdorff space X every nested intersection of compact, connected subsets $\{B_\lambda | \lambda \in \Lambda\}$ is connected.

  PART II

- 9. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$ (these are the homomorphisms $\pi_1(X, x_0) \to \pi_1(X, x_1)$ induced by α and β)

- 10. Let $p : E \to B$ be a covering map. Show that if B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

- 11. Show that if $g : S^1 \to S^1$ is continuous and $g(x)$ is not equal $g(-x)$ for every x, then g is surjective.

- 12. Define a relation on $\mathbb{R}^2$ by $(x_0, y_0) \sim (x_1, y_1)$ if $x_1 - x_0 \in \mathbb{Z}$ and $y_1 - y_0 \in \mathbb{Z}$. Show

  a) This is an equivalence relation.

  b) The quotient map $p: \mathbb{R}^2 \to \mathbb{R}^2/\sim$ is a covering projection.

  c) Describe representatives of all homotopy classes of loops in $\mathbb{R}^2/\sim$ with the quotient topology.

  d) $\pi_1(\mathbb{R}^2/\sim) \cong \mathbb{Z}^2$.

- 13. Let Y be a compact metric space and suppose that $p: X \to Y$ is a covering map. Show that for some $\epsilon > 0$ every epsilon ball $B(x, \epsilon)$ in Y is evenly covered.

Topology Preliminary Examination
January 2010

You may omit two questions from each part.

PART I

1. Let X be a set and $\mathcal{F}$ be a collection of functions from X into the real numbers $\mathbb{R}$. For $x \in X$, $\varepsilon > 0$, and $f_1, ..., f_k \in \mathcal{F}$, define

$$B(x, \varepsilon, f_1, ..., f_k) := \{y : |f_i(x) - f_i(y)| < \varepsilon \text{ for all } i = 1, ..., k\}$$

   (a) Prove that the collection of all such sets $B(x, \varepsilon, f_1, ..., f_k)$ is a basis for a topology $\mathcal{W}$ on X.

   (b) Prove that $\mathcal{W}$ is the coarsest (i.e. smallest) topology on X such that every $f \in \mathcal{F}$ is continuous.

   (c) Show that if X has a completely regular topology $\mathcal{T}$ and $\mathcal{F}$ is the collection of all continuous functions with respect to $\mathcal{T}$ then $\mathcal{T} = \mathcal{W}$.

2. A topological space X is said to be *perfectly normal if* X is normal and every closed subset of X is a G_δ-set (i.e. the intersection of a countable collection of open sets). Prove that every metrizable topological space is perfectly normal.

3. Let X be a compact, connected metric space and $x, y \in X$. Prove that there is a compact, connected subset A of X containing x, y with the following property: If C is any compact, connected subset of X containing x and y then C is not a proper subset of A.

4. Let X be a topological space and $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of connected subsets of X such that the intersection of the closures $\bigcap_{\alpha \in \Lambda} \overline{A}_\alpha$ is non-empty. Let $A := \bigcup_{\alpha \in \Lambda} A_\alpha$.

   (a) Show by example that A need not be connected.

   (b) Prove that A is connected, provided at least one A_α is closed.

5. A Hausdorff space X has the following property: For any closed sets $A, B \subset X$ and open set U in $X \times X$ containing $A \times B$, there are open sets V, W containing A, B, respectively, such that $V \times W \subset U$. Prove that X is normal.

6. Let $\mathcal{A}$ be a collection of open sets in X. Suppose U and V are open sets in X so that $\mathcal{A} \cup \{U \cap V\}$ is a cover of X. Prove that if both $\mathcal{A} \cup \{U\}$ and $\mathcal{A} \cup \{V\}$ have finite subcovers, then $\mathcal{A} \cup \{U \cap V\}$ has a finite subcover.

7. Let X be a compact, Hausdorff topological space and $U \subset X$ be open.

(a) Show that U has a one-point compactification $\widehat{U} = U \cup \{\infty\}$.

(b) Prove that the function $f : X \to \widehat{U}$ that is the identity on U and takes $X \backslash U$ onto ∞ is a quotient map.

8. Let X be a metric space. Prove that X is totally bounded if every sequence in X has a Cauchy subsequence. If you want to use the metric completion of X for the proof, then you must prove here that it exists (not the recommended strategy!).

PART II

9. Compute the fundamental group of the following subset of $\mathbb{R}^3$: The union of the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with the line segment joining the antipodal points $(0, 0, 1)$ and $(0, 0, -1)$. You do not need to explicitly define any deformation retractions that you use, but you must give a convincing description. If necessary you may use the fundamental group of the circle without proving what it is here.

10. Suppose a topological space $X = U \cup V$, U and V are open in X, and $U \cap V$ is path connected with $x_0 \in U \cap V$. Let $i : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j : \pi_1(V, x_0) \to \pi_1(X, x_0)$ be induced by the inclusion maps. Prove that $\pi_1(X, x_0)$ is generated by the images of i and j. That is, each element of $\pi_1(X)$ is a finite product of elements, each of which is in the image of i or in the image of j.

11. Let P denote the projective plane. Prove that any covering map $h : P \to P$ must be a homeomorphism. As part of this problem you should explain, without checking all details, how to compute the fundamental group of P.

12. Consider the set A consisting of the Euclidean plane with the interior of the unit disk removed–that is, $A := \{(x, y) : x^2 + y^2 \geq 1\}$.

    (a) Write down explicitly (i.e. with a formula!) the universal covering map of A, and verify that the map is the universal covering map.

    (b) Use the universal covering map to compute the fundamental group of A.

    (c) Use a deformation retract and the fundamental group of a familiar space (which you do not have to compute here) to compute the fundamental group of A.

13. Let $p : E \to B$ be a covering map. Suppose $f : I \times I \to E$ is a function ($I = [0, 1]$) such that $p \circ f$ is continuous, $f|_{I \times \{0\}}$ is continuous, and $f|_{\{t\} \times I}$ is continuous for all $t \in I$. Show that f is continuous.