ALGEBRA PRELIM EXAM, AUGUST 2024

Instructions: Attempt *all* problems in all parts. Justify your answers.

General assumptions: All rings have $1 \neq 0$ and all modules are unitary.

Part I.

1. Let G be a group of order 204. Suppose that G contains a subgroup H of order 6. Prove that G contains a normal, Abelian subgroup of order 51.
2. Let G be a finite group with H a subgroup. Suppose that the greatest common divisor of $|H|$ and $[G : H]$ is greater than 1. Prove that there exists $g \in G \setminus H$ such that $H \cap gHg^{-1}$ contains a non-identity element.

Part II.

1. Let R be a commutative ring, and suppose that $P_1, \dots, P_n \subset R$ are prime ideals. If the intersection $Q = P_1 \cap \cdots \cap P_n$ is also a prime ideal, then show that $Q = P_i$ for some i.
2. Suppose that R is a Noetherian unique factorization domain. Prove that R is a principal ideal domain if and only if whenever $f, g \in R$ and the gcd of f and g is 1, the ideal (f, g) is equal to R.

Part III.

1. Let R be a principal ideal domain. Suppose that
$$M = R/r_1 \oplus R/r_2 \oplus R/r_3 \oplus R/r_4$$
with $r_1, r_2, r_3, r_4 \in R$ non-zero, not units, and with r_i dividing r_{i+1} for $i = 1, 2, 3$. Prove that there exists a finitely generated R-module N such that $M \cong N \otimes_R N$ if and only if r_1, r_2, and r_3 are associates.
2. Let k be a field, and consider the polynomial ring $R = k[x, y]$. Let I be the ideal (x^2, xy, y^3) in R. We can consider $S = k[x]$ as a subring of R, which makes R/I into an S-module. Write R/I as a direct sum of cyclic S-modules.

Part IV.

1. Let K be a finite field of characteristic p with p^k elements. Suppose that F, L are subfields of K with $|F| = p^n$ and $|L| = p^m$. Also, suppose that $|F \cap L| = p$. Prove that $K = FL$ if and only if $nm = k$.
2. Let E be a finite Galois extension of a field F with Galois group $G = \mathrm{Gal}(E/F)$. Let K be an intermediate field $F \subset K \subset E$ with corresponding subgroup $H = \mathrm{Gal}(E/K) < G$. Prove that the Galois closure of K/F is the subfield corresponding to the subgroup $\cap_{g \in G}\, gHg^{-1}$ in G.

ALGEBRA PRELIMINARY EXAM

JANUARY 2024

Instructions: Attempt *all* problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Prove or give a counterexample: Every group of order 2024 is solvable. (You may use the fact that $2024 = 2^3 \cdot 11 \cdot 23$.)
2. Let p be a prime number and let $\mathbb{F}_p$ be the finite field with p elements. Let $V = \mathbb{F}_p^2$, and recall that $G = \mathrm{GL}_2(\mathbb{F}_p)$ is the group of invertible linear transformations on V. G acts on V in the usual way (by multiplication).
   (a) (2 points) Describe the orbits of this group action.
   (b) (2 points) Describe the stabilizer in G of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   (c) (4 points) Consider now the action of G on $V \times V$ (acting independently on each of the two vectors). How many orbits does this action have?
   (d) (2 points) What is the cardinality of G? Remember to justify your answer.

Part II

1. Let R and S be rings and let 1_R and 1_S denote their respective identities. Let $\varphi : R \to S$ be a nonzero ring homomorphism.
   (a) Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is a zero divisor in S. Deduce that if S is an integral domain then every nonzero ring homomorphism from R to S sends the identity of R to the identity of S.
   (b) Prove that if $\varphi(1_R) = 1_S$ then $\varphi(u)$ is a unit in S and $\varphi(u^{-1}) = \varphi(u)^{-1}$ for each unit u of R.
2. Assume R is commutative and for each $a \in R$ there is an integer $n > 1$ (depending on a) such that $a^n = a$. Prove that every prime ideal of R is a maximal ideal.

Part III

1. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the nonzero element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor (i.e., it cannot be written as $a \otimes b$ for some $a, b \in I$).
2. Let R be a commutative ring (with $1 \neq 0$) and let M be a unitary R-module. Show that $R \otimes_R \mathrm{Hom}_R(R \oplus R, M)$ is a projective R-module if and only if M is a projective R-module.

Part IV

1. Let F be an extension field of the field K with $[F : K] = m$. Let $f(x) \in K[x]$ be irreducible over K and $\deg(f) = n$, where $\gcd(m, n) = 1$. Prove that f is irreducible over F.
2. Let F/K be a finite Galois extension and let E/K be any extension. Prove that FE/E is a Galois extension and has Galois group $\mathrm{Gal}(FE/E) \cong \mathrm{Gal}(F/F \cap E)$.

ALGEBRA PRELIMINARY EXAM

AUGUST 2023

Instructions: Attempt *all* problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Recall that $C(G)$ denotes the center of a group G.
   (a) Let G be a finite group and let N be a normal subgroup such that $N \subseteq C(G)$ and G/N is cyclic. Show that G is abelian.
   (b) Show that every group of order $255 = 3 \cdot 5 \cdot 17$ is abelian.
2. Let G be a finite p-group and let $C(G)$ denote the center of G. Show that if N is a non-trivial normal subgroup of G then $N \cap C(G)$ is a non-trivial normal subgroup of G.

Part II

1. (a) Show that the polynomial $x + 1$ is a unit in the power series ring $\mathbb{Z}[[x]]$, but is not a unit in $\mathbb{Z}[x]$.
   (b) Show that $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.
2. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal I of $\mathbb{Z}[i]$.

Part III

1. Let R be an integral domain. Prove that R is a field if and only if every R-module is projective.
2. Let R be an integral domain and let Q be its field of fractions. If A is an R-module, prove that every element of $Q \otimes_R A$ can be written as a simple tensor $q \otimes a$ for $q \in Q$ and $a \in A$.

Part IV

1. Let F be a field of prime characteristic p. Suppose $E = F(\alpha)$ is a simple extension such that $\alpha \notin F$ but $\alpha^p - \alpha \in F$.
   (a) Find $[E : F]$.
   (b) Prove that E/F is a Galois extension.
   (c) Find the Galois group $\mathrm{Gal}(E/F)$.
   [Hint: Note that $(x + 1)^p - (x + 1) = x^p - x$ in characteristic p.]
2. Let $\zeta := e^{2\pi i/7}$ be a primitive 7th root of unity and consider the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$.
   (a) Prove that there exists an element $\alpha \in \mathbb{Q}(\zeta)$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.
   (b) Express α explicitly as a polynomial in ζ.

ALGEBRA PRELIMINARY EXAMINATION
SPRING 2023

- **Attempt all four parts. Justify your answers.**
- **Note: Rings are assumed to be commutative and with $1 \neq 0$. Modules are assumed to be unitary left modules. $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{F}_q$ denotes a finite field of q elements.**

Part I.

1. Show that if G is a group of order 2023, then G is an Abelian group.
2. Let G be a group of order 3202 and let $C(G)$ denote the center of G. Show that either G is cyclic or $C(G)$ is trivial. (Hint: 1601 is a prime number.)

Part II.

1. Given Principal Ideal Rings A and B, show that the product-ring $A \times B$ is a Principal Ideal Ring.
2. Suppose n is a positive integer and R is a ring with only n (distinct) maximal ideals such that R_M (= localization of R at the maximal ideal M) is a field for each maximal ideal M of R. Show that there are fields $K_1, \ldots, K_n$ such that R is isomorphic (as a ring) to the product-ring $K_1 \times \cdots \times K_n$.

Part III.

1. Let R be a Principal Ideal Domain and let J be a nonzero proper ideal of R. Suppose n is a positive integer and $h : R^n \longrightarrow \bigoplus_{1 \leq m \leq 2n} R/J^m$ is a R-module homomorphism. Show that h is neither injective nor surjective.
2. Let R be an integral domain with quotient-field K and let M be a R-submodule of K. For an integer $n \geq 2$, suppose the n-fold tensor product $M \otimes_R M \otimes_R \cdots \otimes_R M$ is a torsion-free R-module. Then, given a permutation σ of $\{1, 2, \ldots, n\}$ and $x_1, \ldots, x_n \in M$, show that

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \;=\; x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \qquad (\text{in } M \otimes_R M \otimes_R \cdots \otimes_R M).$$

Part IV.

1. Let $K < L$ be fields such that $[L : K] = 2$. Let E be a purely transcendental field-extension of L of finite transcendence degree. If the fixed-field of $G := Aut(E/K)$ is L, then show that L is purely inseparable over K.
2. Let K be a field and let X be an indeterminate. For an integer n, define

$$f_n \;:=\; X^3 - (4n^2 + 2n + 7)\,X - (4n^2 + 2n + 7) \;\in\; K[X]$$

and let $G(n,\ K)$ denote the Galois-group of f_n over K. For each integer n, determine up to isomorphism, the groups $G(n,\ \mathbb{F}_2)$, $G(n,\ \mathbb{Q})$ and $G(n,\ \mathbb{F}_3)$.

ALGEBRA PRELIMINARY EXAMINATION
FALL 2022

- **Attempt all four parts. Justify your answers.**
- **For a positive integer n, the group of permutations (resp. even permutations) of $\{1, \ldots, n\}$ is denoted by S_n (resp. A_n) and $\mathbb{Z}_n$ denotes the additive group of integers modulo n.**
- **Rings are assumed to be commutative with $1 \neq 0$ and modules are assumed to be unitary left modules.**

Part I.

1. Show that a group of order 81522 is solvable but a group of order $8 \times 15 \times 22$ need not be solvable. (Hint: 647 is a prime divisor of 81522.)
2. If a group G of order 2022 has at least 1 but at most 666 elements of order 6, then show that G is cyclic. (Hint: 337 is a prime divisor of 2022.)

Part II.

1. Let R be a ring and $a, b \in R$. For a positive integer n, let $J_n := R\,a^n + R\,b^n$. Show that if J_1 is a principal ideal generated by a non-zerodivisor of R, then J_n is a principal ideal generated by a non-zerodivisor of R for each $n \geq 2$. Find an example of a ring R with elements $a, b \in R$ such that for each $n \geq 2$, J_n is a principal ideal generated by a non-zerodivisor of R but J_1 is not a principal ideal of R.
2. Let R be a Unique Factorization Domain. Suppose R has finitely many irreducibles $p_1, \ldots, p_n$ such that each irreducible element of R is an associate of exactly one of $p_1, \ldots, p_n$. Show that R is a Principal Ideal Domain.

Part III.

1. Let R be a Principal Ideal Domain and suppose M is a finitely generated R-module such that $Hom_R(Hom_R(M,\, R),\, R)$ is R-module isomorphic to M. Show that M is a free R-module.
2. Let V be a vector space over $\mathbb{Q}$. For $v_1, v_2, v_3 \in V$, define
$$f(v_1, v_2, v_3) \;:=\; \sum_{\sigma \in S_3} sgn(\sigma)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \;\in\; V \otimes_{\mathbb{Q}} V \otimes_{\mathbb{Q}} V.$$
Show that $f(v_1, v_2, v_3) = 0$ if and only if v_1, v_2, v_3 are $\mathbb{Q}$-linearly dependent.

Part IV. Let X be an indeterminate.

1. Let $K \leq E$ be fields such that $[E : K] = 2022$ and E/K is Galois. Show the existence of a cubic polynomial $f \in K[X]$ such that f is irreducible in $K[X]$ and has 3 distinct roots in E.
2. Let p be a prime number, let G_p denote the Galois group of $X^6 - p$ over $\mathbb{Q}$ and let
$$\mathfrak{L} := \{S_6,\; A_6,\; S_4 \times S_3,\; \mathbb{Z}_{12},\; S_3 \times S_2,\; \mathbb{Z}_6,\; S_3 \times \mathbb{Z}_6,\; A_3 \times \mathbb{Z}_6,\; \mathbb{Z}_2 \times \mathbb{Z}_6\}.$$
Determine, with proof, the set of all $H \in \mathfrak{L}$ such that H is isomorphic to G_p for some prime p.

Algebra Preliminary Examination

January 2022

Attempt all questions, and justify each answer.

Part I

1. Let G be a group of order $5175 = 3^2 . 5^2 . 23$. Prove that if H is a normal subgroup of order 23 in G, then H is contained in the center of G.

2. Let G be a group of order $2k$, where k is an odd positive integer. For each element $g \in G$ let σ_g denote the permutation $x \mapsto g\,x$ of G, and let $\Gamma = \{\sigma_g \mid g \in G\}$.

(a) Prove that Γ contains an odd permutation.

(b) Prove that G has a subgroup of order k.

Part II

1. Let R be the ring $\mathbb{Z}[\sqrt{2}]$, consisting of all real numbers $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$. Prove that R is a Euclidean domain, with respect to the norm $N(a + b\sqrt{2}) = |\,a^2 - 2b^2\,|$.

2. Let R be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of R is a prime ideal, then R is a field.

Part III

1. Let R be a commutative ring with $1 \neq 0$. It is assumed that for each ideal I of R the quotient ring R/I is given the natural R–module structure $r\,.\,(x + I) = (rx) + I$.

(a) Let I, J be ideals of R. Prove that $R/I \otimes_R R/J$, $R/(I + J)$ are isomorphic as R–modules.

(b) Let M_1, M_2 be distinct maximal ideals of R. Prove that $R/M_1 \otimes_R R/M_2 = 0$.

2. Let R be the polynomial ring $\mathbb{Z}[x]$, and let $I = (2, x)$, the ideal of R generated by the elements $2, x$. Define R–module homomorphisms $\sigma : R \to R \oplus R$, $\tau : R \oplus R \to I$ as follows:
$\sigma(h) = (xh\,,\,-2h)$, $\tau(f\,,\,g) = 2f + xg$.

(a) Prove that $0 \to R \xrightarrow{\sigma} R \oplus R \xrightarrow{\tau} I \to 0$ is a short exact sequence of R–module homomorphisms.

(b) Prove that I is not a projective R–module.

Part IV

In this part, x denotes an indeterminate.

1. Let $f \in \mathbb{Q}[x]$ be irreducible, with splitting field E over $\mathbb{Q}$. Assume that the degree of E over $\mathbb{Q}$ is an odd integer, and that E contains an intermediate field K with $[K : \mathbb{Q}] = 3$. Prove that the irreducible factors of f, considered as a polynomial over K, all have the same degree.
Hint: First show that K is a normal extension of $\mathbb{Q}$.

2. Let G be the Galois group of the polynomial $f = x^4 + 2x^2 + 4 \in \mathbb{Q}[x]$. Determine the order of G, and describe how each element of G permutes the roots of f.

Algebra Preliminary Examination

August 2021

Attempt all questions, and justify each answer.

Part I

1. Let G be a group. Recall that the *commutator subgroup* $[G, G]$ of G is the subgroup generated by all commutators $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ $(g_1, g_2 \in G)$. Also recall that a subgroup H of G is *characteristic in* G, written $H \operatorname{char} G$, if each automorphism of G maps H onto itself.

(a) Define subgroups $G^{(n)}$ $(n \in \mathbb{Z}, n \geq 0)$ inductively as follows:

$$G^{(0)} = G \quad , \quad G^{(n+1)} = [G^{(n)}, G^{(n)}] \ .$$

Prove that $G^{(n)} \operatorname{char} G$ for all $n \geq 0$.

(b) Suppose that G is a non-trivial finite group, such that $G^{(n)} = 1$ for some $n > 0$. Prove that G has a non-trivial characteristic subgroup of prime power order. (*Hint:* consider the subgroup $G^{(n-1)}$, where n is the smallest integer for which $G^{(n)} = 1$.)

2. The *holomorph* of a group G, denoted $\operatorname{Hol}(G)$, is defined to be the semidirect product $G \rtimes_\phi \operatorname{Aut}(G)$, where $\phi : \operatorname{Aut}(G) \to \operatorname{Aut}(G)$ is the identity map. Thus we may identify $\operatorname{Aut}(G)$ with the subgroup $K = \{(1, \sigma) \ : \ \sigma \in \operatorname{Aut}(G)\}$ of the semidirect product $\operatorname{Hol}(G)$. As usual we identify G with the (normal) subgroup $\{(g, 1) \ : \ g \in G\}$ of $\operatorname{Hol}(G)$.

Let $G = \{1, z_1, z_2, z_3\}$ be the non-cyclic group of order 4 (*i.e.* G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). Prove that $\operatorname{Hol}(G)$ is isomorphic to the symmetric group S_4. (*Hint:* Consider the action by left multiplication of $\operatorname{Hol}(G)$ on the four left cosets K, z_1K, z_2K, z_3K of K.)

Part II

1. Let R be an integral domain with the property that every ideal generated by two elements of R is principal.

(a) Prove that every finitely generated ideal of R is principal.

(b) Suppose that R also satisfies the ascending chain condition on principal ideals, *i.e.* given any chain of principal ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$, there exists a positive integer k such that $I_k = I_{k+n}$ for all positive integers n. Prove that R is a principal ideal domain.

2. Recall that an element e of a ring R is *idempotent* if $e^2 = e$. In this question all rings are assumed to be commutative and with $1 \neq 0$.

(a) Let R be a ring containing an idempotent e distinct from $0, 1$. Prove that R is isomorphic to a direct product of two rings. (*Hint:* if e is idempotent, then so is $1 - e$.)

(b) Suppose that R is a finite ring and that every element of R is idempotent. Prove that R is isomorphic to the direct product of finitely many copies of the field with two elements.

Part III $\boxed{\textit{In this part, all } R\textit{–modules } M \textit{ are assumed to be unital, i.e. } 1.m = m \textit{ for all } m \in M\,.}$

1. Recall that given left R–modules D, M, N, an R–module homomorphism $\phi : M \to N$ induces a homomorphism of Abelian groups $\phi' : \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N)$ given by $\phi'(\alpha) = \phi \circ \alpha$.

Let R be a ring with $1 \neq 0$ and let D, L, M, N be left R–modules. Prove that if the sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

of module homomorphisms is exact, then the sequence of induced homomorphisms of Abelian groups

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\phi'} \mathrm{Hom}_R(D, M) \xrightarrow{\psi'} \mathrm{Hom}_R(D, N)$$

is also exact.

2. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$, x being an indeterminate. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ inherits from R a natural R–module structure, with annihilator I.

(a) Show that there is an R–module homomorphism from $I \otimes_R I$ to $\mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\dfrac{p(0)}{2}\, q'(0)$, where q' denotes the usual polynomial derivative of q.

(b) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Part IV $\boxed{\textit{In this part, } x \textit{ denotes an indeterminate.}}$

1. This question concerns the polynomial $f(x) := x^{p^n} - x + 1 \in \mathbb{F}_p[x]$ $(n \geq 1)$. We take some fixed algebraic closure $\mathcal{A}$ of $\mathbb{F}_p$, and denote by $\mathbb{F}_{p^k}$ the unique field of order p^k contained in $\mathcal{A}$. You may assume that each extension of finite degree of $\mathbb{F}_p$ is Galois over $\mathbb{F}_p$, with cyclic Galois group generated by the Frobenius automorphism $\phi : a \mapsto a^p$.

(a) Let E be the splitting field over $\mathbb{F}_p$ of $f(x) = x^{p^n} - x + 1$ in $\mathcal{A}$. Show that E contains $\mathbb{F}_{p^n}$ as a subfield. (*Hint:* If α is a root of $f(x)$, then so is $\alpha + a$ for each $a \in \mathbb{F}_{p^n}$.)

(b) Determine the order of the Frobenius automorphism $\phi : E \to E$, $\phi : \beta \mapsto \beta^p$. (*Hint:* First compute $\phi^n(\alpha)$, where α is a root of $f(x)$.)

(c) Show that if $f(x)$ is irreducible over $\mathbb{F}_p$, then $pn = p^n$.
[*Observation (you may omit the easy proof):* from $pn = p^n$ it follows that $n = 1$ or $n = p = 2$.]

2. Determine the Galois group over $\mathbb{Q}$ of $x^4 + 9$, describing how each automorphism permutes the roots of this polynomial.

Algebra Preliminary Examination

January 2021

Attempt all questions, and justify each answer.

Part I

1. Let p be a prime, and let S_p denote the symmetric group of degree p. Prove that if P is a subgroup of S_p of order p, then the normalizer of P in S_p has order $p(p-1)$.

2. Classify, up to isomorphism, the groups of order 63.

Part II

1. A *local ring* is a commutative ring with $1 \neq 0$ that has a unique maximal ideal. Prove that if R is a local ring with maximal ideal M, then every element of $R \setminus M$ is a unit. Also prove that if R is a commutative ring with $1 \neq 0$, in which the set of nonunits forms an ideal M, then R is a local ring with maximal ideal M.

2. Let $p \in \mathbb{Z}_+$ be prime, and let $\mathbb{Z}[i]$ denote the usual ring of Gaussian integers $\{ a+bi \mid a, b \in \mathbb{Z} \}$. For which p is the quotient ring $\mathbb{Z}[i]/(p)$ (i) a field, (ii) a product of fields? Justify your answer. (You may use the following facts: (i) $\mathbb{Z}[i]$ is a Euclidean Domain with respect to the field norm, hence is also a Unique Factorization Domain, and (ii) a prime $p \in \mathbb{Z}_+$ with $p \equiv 1 \pmod 4$ can be written as the sum of two integer squares.)

Hint: Use the Chinese Remainder Theorem where appropriate. Also note that a product of fields cannot contain a nonzero nilpotent element.

Part III

1. Let V be a finite dimensional vector space over a field F, and let v_1, v_2 be nonzero elements of V. Prove that $v_1 \otimes v_2 = v_2 \otimes v_1$ in $V \otimes_F V$ if and only if $v_1 = \lambda v_2$ for some $\lambda \in F$.

2. Let R be a ring with $1 \neq 0$, let P, M, N be R–modules, and let there be an exact sequence of R–module homomorphisms $M \xrightarrow{\phi} N \to 0$.

(a) Prove that if P is a direct summand of a free R–module, then the induced sequence of Abelian group homomorphisms

$$\mathrm{Hom}_R(P, M) \xrightarrow{\phi'} \mathrm{Hom}_R(P, N) \to 0$$

is exact. *(Here ϕ' is the homomorphism $\psi \mapsto \phi \circ \psi$.)*

(b) Show by means of an example that in general the induced sequence $\mathrm{Hom}_R(P, M) \xrightarrow{\phi'} \mathrm{Hom}_R(P, N) \to 0$ need not be exact.

Note: For this question do not assume any result concerning projective modules.

Part IV $\boxed{\textit{In this part, } x \textit{ denotes an indeterminate.}}$

1. This question concerns the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

(a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$, and that its roots in $\mathbb{C}$ are $\pm\alpha$, $\pm\beta$, where $\alpha = \sqrt{1+\sqrt{3}}$, $\beta = \sqrt{1-\sqrt{3}}$.

(b) Prove that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, and that $[\mathbb{Q}(\alpha,\beta) : \mathbb{Q}(\alpha)] = 2$.

(c) Prove that the splitting field of $x^4 - 2x^2 - 2$ has degree 8 over $\mathbb{Q}$, and that the Galois group of this polynomial over $\mathbb{Q}$ is dihedral of order 8 .

Hint for (c): The Galois group acts faithfully on the set of roots of the polynomial.

2. Let $\mathbb{F}_p$ denote the field of order p , let $f \in \mathbb{F}_p[x]$ be irreducible over $\mathbb{F}_p$, and let K be a splitting field for f over $\mathbb{F}_p$.

Let L be an intermediate field, *i.e.* $\mathbb{F}_p \subseteq L \subseteq K$. Prove that the irreducible factors of the polynomial f in $L[x]$ all have the same degree.

Hint: Here is one approach. Let $g \in L[x]$ be a factor of f that is irreducible in $L[x]$, and let α be a root of g in K . Consider the relationship between $[L(\alpha) : L]$ and $[K : L]$.

Algebra Preliminary Examination

August 2020

Attempt all questions, and justify each answer.

Part I

1. Let P be a Sylow p–subgroup of a finite group G . If p is the smallest prime dividing $|G|$ and P is cyclic, prove that $N_G(P) = C_G(P)$. (Recall that $N_G(P)$, $C_G(P)$ denote the normalizer and centralizer of P in G , respectively.)

(*Hint:* Consider the order of the automorphism group of P and the action of $N_G(P)$ on P by conjugation.)

2. **(a)** Prove that a group of order 105 contains a cyclic normal subgroup of order 35 .

(b) Prove that, up to isomorphism, there is just one non-Abelian group of order 105 .

In parts II, III and IV, X denotes an indeterminate.

Part II

1. Let R be a commutative ring with $1 \neq 0$. Recall that R is *Artinian* if it satisfies the descending chain condition on ideals, *i.e.* if $I_1 \supseteq I_2 \supseteq \ldots$ is a descending chain of ideals of R , then there exists $k \in \mathbb{Z}_+$ such that $I_m = I_k$ for all $m > k$.

Let S be an arbitrary commutative ring with $1 \neq 0$, and let J denote the Jacobson radical of $S[X]$. Prove that $S[X]/J$ is not Artinian.

2. Let R be the subring of $\mathbb{Q}[X]$ consisting of all polynomials whose constant term is an integer.

(a) Prove that R is an integral domain in which every irreducible element is prime.

(b) Prove that R is not a Unique Factorization Domain.
(*Hint:* Consider factorizations of the element X .)

Part III

1. Let k be a field, and let $R = M_2(k)$ be the ring of 2×2 matrices over k . Let P be the set of 2×1 matrices over k : then P is an Abelian group under matrix addition, and left matrix multiplication of elements of P by elements of R accords P the structure of a left R–module.

Prove that the R–module P is projective, but not free.

2. Let $R = \mathbb{Z}[X]$, let $I \subset R$ be the ideal generated by $2, X$, and let $M = I \otimes_R I$.

Prove that the element $2 \otimes 2 + X \otimes X \in M$ cannot be written as a simple tensor $a \otimes b$ $(a , b \in I)$.

(*Hint:* Use a suitable R–module homomorphism defined on M .)

Part IV

1. Prove that $\mathbb{Q}(\sqrt{5+2\sqrt{5}})$ is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

2. Let F be a field (possibly infinite) of finite characteristic p, and let $a \in F$ be an element not of form $b^p - b$ for any $b \in F$. Let $f = X^p - X - a \in F[X]$.

(a) Prove that the polynomial f is separable and irreducible over F.

(b) Prove that if α is a root of f in an extension field of F, then $F(\alpha)$ is a splitting field for f over F.

(*Hint:* Consider the effect of substituting $X+1$ for X in the polynomial f.)

ALGEBRA PRELIMINARY EXAM

JANUARY 2020

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a finite group and $\phi : G \to H$ a *surjective* homomorphism. Prove that if $y \in H$ is such that $|y| = p^r$, for some prime p and $r \in \mathbb{Z}_{>0}$, then there is $x \in G$ such that $\phi(x) = y$ and $|x| = p^s$, for some $s \in \mathbb{Z}_{>0}$.

   [**Hint:** Let $g \in G$ such that $\phi(g) = y$, and write $|g| = n \cdot p^k$, where $p \nmid n$.]

2. Let G be a group of order 60 and assume that 4 divides $|Z(G)|$ [where $Z(G)$ denotes the *center* of G]. Prove that G must be Abelian.

Part II

1. Let I be the ideal of $\mathbb{Z}[x]$ generated by 7 and $x^2 + 1$. Prove that I is a maximal ideal.

2. Let R be an *integral domain* such that for any descending chain of ideals
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$
   there is a positive integer N such that $I_i = I_N$ for all $i \geq N$. Prove that R is a field.

Part III

1. Let R be a subring of S. Prove that $S \otimes_R S \neq 0$.

2. Let R be a ring containing $\mathbb{Z}$ such that R is a free $\mathbb{Z}$-module of finite rank $n > 0$ and every non-zero ideal of R has a non-zero element of $\mathbb{Z}$. Prove that for every non-zero ideal I we have that R/I is finite.

Part IV

1. Given a prime p and a positive integer n, show that there is an *Abelian* extension [i.e., Galois with Abelian Galois group] K of $\mathbb{Q}$ with $[K : \mathbb{Q}] = p^n$.

2. Let F be a field of characteristic p with exactly p^r elements. If K is a finite extension of F with $K = F[\alpha]$, for some $\alpha \in K$, and f is the minimal polynomial of α over F, then show that if β is another root of f, then $\beta \in K$ and $\beta = \alpha^{p^k}$ for some $k \in \mathbb{Z}$.

ALGEBRA PRELIMINARY EXAM

AUGUST 2019

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G_1, G_2 be groups, $N \trianglelefteq G_1$, and $\phi : G_1 \to G_2$ be an onto homomorphism such that $N \cap \ker(\phi) = \{1\}$. Prove that for $x \in N$ we have that $\mathrm{C}_{G_2}(\phi(x)) = \phi(\mathrm{C}_{G_1}(x))$. [Remember: $\mathrm{C}_G(x) \stackrel{\text{def}}{=} \{g \in G : gx = xg\}$ is the *centralizer* of x in G.]

2. Let G be a group of order $992 = 2^5 \cdot 31$. Prove that either G has a normal subgroup of order $32 = 2^5$ or it has a normal subgroup of order 62.

Part II

1. Let R be a UFD with exactly two non-associate prime elements p and q [i.e., p and q are non-associate primes and every prime is an associate of either p or q]. Prove that R is a PID.

2. Let R be a PID and P a prime ideal of $R[x]$ such that $P \cap R \neq \{0\}$. Prove that there is $p \in R$ prime [in R] such that either $P = (p)$ or $P = (p, f)$ for some f prime in $R[x]$.

Part III

1. Let R be a commutative ring and M an R-module. Prove that $R \otimes_R \mathrm{Hom}_R(R \oplus R, M)$ is projective if and only if M is projective.

2. Let R be a commutative ring, M and N be R-modules and M' and N' be submodules of M and N respectively. Define L as the submodule of $M \otimes_R N$ generated by the set
$$\{x \otimes y \in M \otimes_R N \, : \, x \in M' \text{ or } y \in N'\}.$$
Show that $M/M' \otimes_R N/N' \cong (M \otimes_R N)/L$.

Part IV

1. Let $F = \mathbb{Q}(\sqrt[3]{2} \cdot \zeta)$, where $\zeta = -1/2 + \sqrt{3}\mathrm{i}/2$ [a primitive third root of unity]. Prove that -1 is not a sum of squares in F, i.e., there is no positive integer n and $\alpha_1, \ldots, \alpha_n \in F$ such that $-1 = \alpha_1^2 + \cdots + \alpha_n^2$.

2. Let F be a field of characteristic 0 and K/F be a field extension of degree n such that there is a root of unity ζ in the algebraic closure of K such that $K \subseteq F[\zeta]$. Prove that if $d \mid n$, there is $\alpha \in K$ such that the minimal polynomial of α over F has degree d.

ALGEBRA PRELIMINARY EXAM

AUGUST 2018

Instructions: Attempt *all* problems in all four parts. Justify your answers.

General assumptions: All rings have $1 \neq 0$, their subrings contain 1, and all modules are unitary.

Part I

1. Let G be a (possibly infinite) group, and suppose that G contains a subgroup $H \neq G$ whose index $[G : H]$ is finite. Prove that G contains a normal subgroup $N \neq G$ of finite index.
2. Prove that every group of order 70 contains a cyclic, normal subgroup of order 35.

Part II

1. Let R be a commutative ring in which every element is either a unit or nilpotent. Prove that R has exactly one prime ideal.
2. If R is an integral domain, prove that there are infinitely many ideals in $R[x]$ that are both prime and principal.

Part III

1. Let R be a ring, possibly non-commutative, and suppose that
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence of left R-modules, with M' and M'' finitely generated. Prove that M is finitely generated.
2. Let M be a finitely-generated $\mathbb{Z}$-module, and let $T \subset M$ be its torsion submodule. Show that the torsion submodule of $M \otimes_{\mathbb{Z}} M$ has at least $|T|$ elements.

Part IV

1. Let p be a prime and suppose that $f \in \mathbb{F}_p[x]$ is an irreducible polynomial. Let K be a degree 2 extension of $\mathbb{F}_p$ and suppose that there exist non-constant polynomials $g, h \in K[x]$ such that $f = gh$. If g is an irreducible polynomial of degree 5, what is the degree of f?
2. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial, and $K/\mathbb{Q}$ is an extension such that f has exactly one root in K. Let G be the Galois group of f, and show that $|G|$ is divisible by 12.

ALGEBRA PRELIMINARY EXAM

AUGUST 2017

Instructions: Attempt *all* problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Suppose that H is a subgroup of a finite group G of index p, where p is the smallest prime dividing the order of G. Prove that H is normal in G.
2. Show that every group of order 222 is solvable.
   Fun fact: The University of Tennessee was established 222 years ago.

Part II

1. Let I and J be ideals of a ring R and assume that P is a prime ideal of R that contains $I \cap J$. Prove that either I or J is contained in P.
2. Let R be an integral domain and suppose that every prime ideal in R is principal. Prove that R is a PID.

Part III

1. Let V be a Noetherian right R-module, and $\theta : V \to V$ a homomorphism.
   (a) Show that $\ker(\theta^{n+1}) = \ker(\theta^n)$ for some $n \geq 1$.
   (b) If θ is onto, show that it is one-to-one.
2. An R-**projection** is defined to be an R-module homomorphism $\varphi : R^n \to R^n$ such that $\varphi^2 = \varphi$. Prove that a finitely generated R-module M is projective if and only if it is isomorphic to the image of some R-projection.

Part IV

1. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F(\alpha)$. Assume that some power of α lies in F and let n be the smallest positive integer such that $\alpha^n \in F$.
   (a) If $\alpha^m \in F$ with $m > 0$, show that m is a multiple of n.
   (b) If E is a separable extension of F, prove that the characteristic of F does not divide n.
   (c) If every root of unity of E lies in F, show that $[E : F] = n$.
2. Let F be a field of characteristic 0 and let E be a finite Galois extension of F.
   (a) If $0 \neq \alpha \in E$ with $E = F(\alpha)$, show that $F(\alpha^2) \neq E$ if and only if there exists $\sigma \in \mathrm{Gal}(E/F)$ with $\sigma(\alpha) = -\alpha$.
   (b) Prove that there exists an element $\alpha \in E$ with $E = F(\alpha^2)$.

ALGEBRA PRELIMINARY EXAMINATION
SPRING 2017

- **Attempt all four parts. Justify your answers.**

Part I.

1. Show that a group of order 255 is not a simple group.
2. A group G has a cyclic normal subgroup of order 2016. If G also has a subgroup of order 2017, then show that G has a cyclic subgroup of order $(2016) \times (2017)$.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$.

1. Let A and B be rings. Show that each ideal of $A \times B$ is of the form $I \times J$, where I is an ideal of A and J is an ideal of B.
2. Let R be a ring, let X be an indeterminate and let $S := \{X^n \mid 0 \leq n \in \mathbb{Z}\}$. If $S^{-1}R[[X]]$ is a field, then show that R is a field.

Part III.

Note : *Rings are assumed to be commutative with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let A be a ring and let M, N be finitely generated projective (left) A-modules. Show that $Hom_A(M, N)$ is a finitely generated projective A-module.
2. Let R be a PID and let I, J be ideals of R. If $I \neq R \neq J$, then show that $(R/I) \oplus (R/J)$ and $(R/I) \otimes_R (R/J)$ are not isomorphic as (left) R-modules.

Part IV.

Note : *In what follows,* X *is an indeterminate.*

1. Let K be an extension-field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}_{30}$. Suppose each of $f, g \in \mathbb{Q}[X]$ is an irreducible polynomial of degree 6 and f has a root $a \in K$. If g has a root in K, then show that g has all its roots in $\mathbb{Q}[a]$.
2. Let $F \subset K$ be finite fields of characteristic 5 and suppose $g \in F[x]$ is irreducible in $F[x]$. If g has degree 11, then show that either g is irreducible in $K[x]$ or all its roots are in K.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2016

- **Attempt all four parts. Justify your answers.**

Part I.

1. Let p be a prime number and G be a non-Abelian group of order p^3. Show that G has at least 3 (distinct) subgroups of index p.
2. Let G be a group of order p^3q, where p, q are distinct prime numbers. If no Sylow p-subgroup of G is normal and also no Sylow q-subgroup of G is normal, then show that G has order 24.

Part II.

<u>**Note**</u>: *Rings are tacitly assumed to be commutative and with* $1 \neq 0$.

1. Let R be a ring, X an indeterminate and $h : R[X] \to R[[X]]$ a ring-homomorphism such that $h(a) = a$ for all $a \in R$. Show that h is not surjective.
2. Let R be an integral domain with at least 3 (distinct) maximal ideals. Given maximal ideals M and N of R, show that $R_M \cap R_N \neq R$. (Here localization of R at a prime ideal is naturally identified as a ring in between R and the quotient-field of R.)

Part III.

<u>**Note**</u>: *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be a ring and let $a \in R$ be a nonzero element of R such that $a^3 = a$. Show that the ideal Ra is a projective R-module.
2. Let R be a PID and let M be a finitely generated R-module. For a maximal ideal Q of R, let $\delta(Q, M)$ denote the dimension of $M \otimes_R R/Q$ as a vector-space over the field R/Q. Let $\delta(M)$ denote the $\sup\{\delta(Q, M)\}$, where the supremum is taken over all maximal ideals Q of R. Show that as an R-module, M has a generating set of cardinality $\delta(M)$ and any generating set of M has cardinality at least $\delta(M)$.

Part IV.

<u>**Note**</u>: *In what follows,* X *is an indeterminate.*

1. Let $f(X)$ be a monic polynomial with rational coefficients. Assume $f(X)$ is irreducible in $\mathbb{Q}[X]$ and the Galois-group of $f(X)$ over $\mathbb{Q}$ is a group of order 99. What is the degree of $f(X)$?
2. Compute the Galois group of $X^6 - 9$ over $\mathbb{Q}$.

ALGEBRA PRELIMINARY EXAM

JANUARY 2016

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a finite group and H be a subgroup of G. Prove that $n_p(H) \leq n_p(G)$, where $n_p(X)$ denotes the number of Sylow p-subgroups of X.

2. Let G be a group of order p^n for some prime p and positive integer n. Prove that if $1 \neq H \trianglelefteq G$, then $Z(G) \cap H \neq 1$. [Here $Z(G)$ denotes the center of G.]

Part II

1. Let R be a *Boolean ring*, i.e., a ring [with 1] for which $a^2 = a$ for all $a \in R$. [You can use without proof the well known fact that if R is Boolean, then it is commutative of characteristic 2.]

(a) Prove that if R is finite, then its order is a power of 2.

(b) Prove that every prime ideal of R is maximal.

2. Show that $R \stackrel{\text{def}}{=} \mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1x_2, x_3x_4, x_5x_6, \ldots)$ has infinitely many distinct *minimal* prime ideals. [P is a minimal prime ideal if it is prime and whenever $Q \subseteq P$, with Q also prime, we have $Q = P$.]

Part III

1. Let F be a field and M be a *torsion* $F[x]$-modulo. Prove that if there is $m_0 \in M$, with $m_0 \neq 0$, and an *irreducible* $f \in F[x]$ such that $f \cdot m_0 = 0$, then $\mathrm{Ann}(M) \subseteq (f)$.

2. Let R be an integral domain and I a principal ideal of R. Prove that $I \otimes_R I$ has no non-zero torsion element [i.e., if $m \in I \otimes_R I$, with $m \neq 0$, and $r \in R$ with $rm = 0$, then $r = 0$].

Part IV

1. Let K/F be an algebraic field extension and $\mathrm{Emb}(K/F)$ denote the set of field homomorphisms $\sigma : K \to \bar{K}$ such that $\sigma(a) = a$ for all $a \in F$. [Here $\bar{K}$ is a fixed algebraic closure of K.]

(a) Prove that if α is a root of a [not necessarily irreducible] non-zero polynomial $f \in F[x]$ with $\deg(f) = n$, then $\mathrm{Emb}(F[\alpha]/F)$ has at most n elements.

(b) Give an example of an algebraic extension K/F of degree greater than one for which $\mathrm{Emb}(K/F)$ has a single element.

2. Let $F = \mathbb{Q}[\sqrt{2}]$ and $K = \mathbb{Q}[\sqrt[8]{2}, \mathrm{i}]$.

(a) Prove that K/F is Galois with $[K : F] = 8$.

(b) Prove that $\mathrm{Gal}(K/F)$ has a non-normal subgroup. [This implies that it is the dihedral group of order 8, as it is the only group of order 8 with this property.]

ALGEBRA PRELIMINARY EXAM

AUGUST 2015

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G be a *non-Abelian* group of order p^3, $[G,G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle$ be its commutator subgroup and $Z(G)$ be its center. Show that $|Z(G)| = p$ and that $Z(G) = [G,G]$.

2. Let G_1 and G_2 be groups of order 81 acting *faithfully* [i.e., only 1 acts as the identity function] on sets X_1 and X_2, respectively, with 9 elements each. Show that there is an isomorphism $\psi : G_1 \to G_2$.

Part II

1. Let D be a *finite* division ring. Prove that D has a prime power number of elements. [**Hint:** Consider the center $Z(D) = \{a \in D : ax = xa \text{ for all } x \in D\}$.]

2. Let $p \in \mathbb{Z}$ prime and
$$f = a_{2n+1}x^{2n+1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x].$$
Prove that if $p^3 \nmid a_0$, $p^2 \mid a_0, a_1, \ldots, a_n$, $p \mid a_{n+1}, a_{n+2}, \ldots, a_{2n}$ and $p \nmid a_{2n+1}$, then f is irreducible in $\mathbb{Q}[x]$.

Part III

1. Let R be a commutative ring. An R-module is *Artinian* if it satisfies the *descending chain condition for submodules.* [I.e., if $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ is a chain of submodules, then there is a i_0 such that for all $i \geq i_0$, we have $S_i = S_{i_0}$.] Show that if L and N are Artinian R-modules and we have a short exact sequence
$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0,$$
then M is also Artinian.

2. Let R be a commutative ring such that every R-module is free. Prove that R is a field.

Part IV

1. Let $\mathbb{F}_p$ be the field with p elements, and t be an indeterminate. Let $f(t), g(t) \in \mathbb{F}_p[t] \setminus \{0\}$, with $\max\{\deg f, \deg g\} < p$ and $f(t)/g(t) \notin \mathbb{F}_p$. Show that the extension $\mathbb{F}_p(t)/\mathbb{F}_p(f(t)/g(t))$ is separable.

2. Suppose that $f = \prod_{i=1}^{N}(x - \alpha_i) \in \mathbb{Q}[x]$ [with $\alpha_i \in \mathbb{C}$] is *irreducible* in $\mathbb{Q}[x]$ and let $f_n \stackrel{\text{def}}{=} \prod_{i=1}^{N}(x - \alpha_i^n)$. Prove that for each n, there is $g_n \in \mathbb{Q}[x]$ *irreducible* and a positive integer k_n such that $f_n = g_n^{k_n}$.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2014

Attempt all four parts. Justify your answers.

Part I.

1. Show that S_4 (the group of permutations of $\{1,2,3,4\}$) does not have a subgroup isomorphic to Q_8 (the quaternion-group of order 8).
2. Let G be a group of order 2014. Show that G is cyclic if and only if G has a normal subgroup of order 2.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$*, subrings are assumed to contain* 1 *and ring-homomorphisms are assumed to map* 1 *to* 1.

1. Let R be an integral domain with only finitely many units. Show that the intersection of all maximal ideals of R is 0.
2. Let R be a ring such that each non-unit of R is nilpotent. Let X be an indeterminate and let $f \in R[[X]]$. Show that $f^n = f$ for some integer $n \geq 2$ if and only if either $f = 0$ or $f^{n-1} = 1$.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let L be a module over a ring R and let M, N be R-submodules of L. Show that if $(M+N)/(M \cap N)$ is a projective R-module then $M/(M \cap N)$ is also a projective R-module.
2. Let R be a PID with infinitely many prime ideals and let M be a finitely generated R-module. Show that M is a torsion R-module if and only if $M \otimes_R R/P = 0$ for all but finitely many prime ideals P of R.

Part IV.

Note : *In what follows,* X *is.an indeterminate.*

1. Let $f(X) := X^5 + 3X^3 + X^2 + 3 \in \mathbb{Q}[X]$. Let K be the splitting field of $f(X)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.
2. Let $f(X) := X^3 + X + 1 \in \mathbb{Q}[X]$. Let F be a finite Galois extension of $\mathbb{Q}$ such that the Galois group of F over $\mathbb{Q}$ is an Abelian group. Show that f is irreducible in $F[X]$.

Algebra Preliminary Exam   January 2014

Attempt all problems and justify all your answers. All rings have a $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I. Groups

1. Show that every group of order 1,225 is abelian.
2. Let $n \geqslant 2$. Show that there is a nontrivial homomorphism $f : S_n \to \mathbb{Z}/n\mathbb{Z}$ (i.e., $\ker f \neq S_n$) if and only if n is even.

Part II. Rings

1. Let R be a commutative ring. Show that $J(R[X]) = nil(R[X])$. ($J(A)$ and $nil(A)$ are the Jacobson and nil radicals of A.)
2. Let R be a PID.
   (a) Show that R satisfies ACC on ideals.
   (b) Show that every nonzero prime ideal of R is maximal.

Part III. Modules

1. Let R be a ring and M a nonzero R-module. Show that $M = A \oplus B$ for proper submodules A and B of M if and only if there is a nonzero, nonidentity homomorphism $f : M \to M$ with $f^2 = f$.
2. Let R be a commutative ring, I a proper ideal of R, and M an R-module. Show that $(R/I) \otimes_R M$ and M/IM are isomorphic as R-modules.

Part IV. Fields

1. Let K a subfield of a field F. Show that there is a subring of F containing K that is a PID, but not a field, if and only if the extension F/K is not algebraic.
2. Determine the Galois group of $f(X) = X^{10} + X^8 + X^6 + X^2$ over $\mathbb{Z}/2\mathbb{Z}$.

Algebra Preliminary Exam

August 2013

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I.

1. (a) Let p and q be (not necessarily distinct) prime numbers. Show that a group G with $|G| = pq$ is either abelian or $Z(G) = \{e\}$.

   (b) Give an example of a nonabelian group G whose order is the product of three (not necessarily distinct) primes and $Z(G) \neq \{e\}$.

2. (a) Let G be a group with $|G| = 100$. Show that G is abelian if and only if its Sylow 2-subgroup is normal.

   (b) Give an example of a nonabelian group of order 100.

Part II.

1. Let R and S be a commutative rings with $1 \neq 0$. Show that every ideal of $R \times S$ has the form $I \times J$ for I an ideal of R and J an ideal of S.

2. Let R be a commutative ring with $1 \neq 0$. Show that $f(X) = a_0 + a_1 X + \cdots + a_n X^n$ is a unit in $R[X]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent.

Part III

1. Let P and Q be finitely generated projective R-modules over a commutative ring R with $1 \neq 0$. Show that $\mathrm{Hom}_R(P,Q)$ is a finitely generated projective R-module.

2. Let R be a commutative ring with $1 \neq 0$, S a nonempty multiplicatively closed subset of R, and M an R-module. Show that $(S^{-1}R)\otimes_R M$ and $S^{-1}M$ are isomorphic as $S^{-1}R$-modules.

Part IV.

1. Let p and q be distinct prime numbers, F a subfield of a field K, and $f(X)$, $g(X) \in F[X]$ be irreducible with $\deg(f(X)) = p$ and $\deg(g(X)) = q$. Let $a, b \in K$ be roots of $f(x)$ and $g(X)$, respectively. Show that $[F(a,b):F] = pq$.

2. (a) Let F be a splitting field for $f(X) \in \mathbb{Q}[X]$ over $\mathbb{Q}$ with abelian Galois group G. Show that every subfield L of F is a splitting field over $\mathbb{Q}$ for some polynomial $g(X) \in \mathbb{Q}[X]$.

   (b) Give an example to show that if G is not abelian in part (a), then some L need not be a splitting field.

ALGEBRA PRELIMINARY EXAM

JANUARY 2013

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let p and q be prime numbers such that $q < p$ and q does not divide $p^2 - 1$. Prove that every group of order p^2q is Abelian.

2. Let G be a finite *simple* group. Show that if p is the *largest* prime dividing $|G|$, then there is no subgroup $H \leq G$ such that $1 < |G : H| < p$.

Part II

1. Let R be a ring not necessarily having 1 [or commutative], with at least two elements and such that for all non-zero $a \in R$ there is a *unique* $b \in R$ such that $aba = a$.
   (a) Show that R has no [non-zero] zero divisors.
   (b) Show that for a and b as above, we also have $bab = b$.
   (c) Show that R has 1.

2. Let R be a commutative ring and $a \in R$ such that $a^n \neq 0$ for all positive integers n. Let I be an ideal maximal with respect to the property that $a^n \notin I$ for any positive integer n. Show that I is prime.

Part III

1. Let $V = \mathbb{R}^2$ and $\{e_1, e_2\}$ be a basis of V. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_{\mathbb{R}} V$ cannot be written as a single tensor.

2. Let R be a PID.
   (a) Prove that a finitely generated R-module M is free if and only if it is torsion free.
   (b) Prove that if a finitely generated R-module M is projective, then it is free.

Part IV

1. Let $\mathbb{F}_p$ be the field with p elements, $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of $\mathbb{F}_p$ and let
$$L = \{\alpha \in \bar{\mathbb{F}}_p \ : \ p \nmid [\mathbb{F}_p[\alpha] : \mathbb{F}_p]\}.$$
   Show that L is a field.

2. Let p be a prime, F be a field of characteristic different from p and $f = x^p - a \in F[x]$ [not necessarily irreducible]. Let K be the splitting field of $x^p - 1$ over F and assume that all roots of f lie in K.
   (a) Show that if $f(\alpha) = 0$ with $\alpha \notin F$, then $F[\alpha] = K$.
   (b) Prove that f has a root in F.

ALGEBRA PRELIMINARY EXAM

AUGUST 2012

Instructions: Attempt *all* problems in all four parts. Justify each answer.

General Assumptions: All rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let G and H be finite Abelian groups. Prove that if $G \times H \times H \cong G \times G \times H$, then $G \cong H$.

2. Let p be a prime and G be a group of order p^n. For $k \in \{1, 2, 3, \ldots, (n-1)\}$, let s_k and n_k denote the number of subgroups and normal subgroups of G of order p^k respectively. Show that $s_k - n_k$ is divisible by p.

Part II

1. Let R be a commutative ring for which every proper ideal is prime. Show that R is a field.

2. Let F be a field and consider the subring R of $F[t]$ given by polynomials with the coefficient of t equal to zero, i.e., $R = F + t^2F[t]$.

(a) Show that R has an irreducible element which is not prime. [Hence, R is not PID.]

(b) Show that R is Noetherian. [**Hint:** Consider a connection between R and $F[x, y]$.]

Part III

1. Let R be a commutative ring, S be a subring of R, A be an R-module and

$$\mathcal{H} \stackrel{\text{def}}{=} \operatorname{Hom}_R(R \otimes_S (S \oplus S), A).$$

Show that for every *surjective* homomorphism of R-modules $\phi : M \to N$ and R-module homomorphism $f : \mathcal{H} \to N$ there is an R-module homomorphism $F : \mathcal{H} \to M$ such that $\phi \circ F = f$ if and only if the same if true if we replace $\mathcal{H}$ by A.

2. Let R be a commutative ring, D, M and N be R-modules, $\phi : M \to N$ be an R-module homomorphism and $1 \otimes \phi : D \otimes_R M \to D \otimes_R N$ be the homomorphism for which

$$(1 \otimes \phi)(d \otimes m) = d \otimes \phi(m).$$

(a) Assume that ϕ is injective. Show that if D is free and of finite rank, then $1 \otimes \phi$ is also injective. [The finite rank is not necessary, but we assume it here for simplicity.]

(b) Show that the above statement is not true for an arbitrary D.

Part IV

1. Let F be a field and K/F be an algebraic extension. Show that if R is a *subring* of K with $F \subseteq R \subseteq K$, then R is a field.

2. Let F be a field, K/F be a Galois extension and $f \in F[x]$ be monic, separable and irreducible. Show that if $f = f_1 \cdots f_k$ is the factorization of f in $K[x]$, with f_i irreducible and monic, then the f_i's are distinct, of the same degree and $G \stackrel{\text{def}}{=} \operatorname{Gal}(K/F)$ acts *transitively* on $\{f_1, \ldots, f_k\}$. [I.e., given $\sigma \in G$, the map $f_i \mapsto f_i^\sigma$ is a permutation of the f_i's and given $i, j \in \{1, \ldots, k\}$, there is a $\tau \in G$ such that $f_i^\tau = f_j$.]

ALGEBRA PRELIMINARY EXAMINATION
Spring 2012

Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 455 is necessarily cyclic.
2. Let G be a group of order 56. Show that G is solvable.

Part II.

1. Let $f : \mathbb{Q} \to \mathbb{Z}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$. Show that the image of f has at most three elements and there exist an infinite number of such functions whose image has three elements.
2. Let R be a PID and let J denote the intersection of all maximal ideals of R. If $a^2 - a$ is in J for all $a \in R$, then show that R has only finitely many maximal ideals.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be an integral domain and let M, N be projective R-modules. Show that $M \otimes_R N$ is a projective R-module.
2. Suppose R is a principal ideal domain that is not a field. Suppose M is a finitely generated R-module such that for every maximal ideal P of R, M/PM is a cyclic R/P-module. Show that M itself is cyclic.

Part IV.

1. Let $f(X)$ be a monic polynomial of degree 9 having rational coefficients. Assume that $f(X)$ is irreducible in $\mathbb{Q}[X]$. Let K denote the splitting field of f over $\mathbb{Q}$ and let $u \in K$ be a root of f. If $[K : \mathbb{Q}] = 27$, then show that $\mathbb{Q}[u]$ has a subfield L with $[L : \mathbb{Q}] = 3$.
2. Let F, K be fields such that K is a finite Galois extension of F with Galois group G. Suppose $a \in K$ is such that $\sigma(a) - a \in F$ for all $\sigma \in G$. If the characteristic of F does not divide the order of G, then show that $a \in F$. Assuming F to be the field of two elements, construct a quadratic field extension $K := F[a]$ of F such that $\sigma(a) - a \in F$ for all $\sigma \in G$.

Algebra Preliminary Exam **January 2011**

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let G be a finite simple group. Show that if G has a subgroup H with $[G:H] = n \geq 2$, then $|H| \mid (n-1)!$.

2. List, up to isomorphism, all groups of order 153. Justify your answer.

II. 1. Let R be a commutative ring and I an ideal of R. Let $I^* = (I, X)$ be an ideal of the polynomial ring $R[X]$. Determine, in terms of I, when I^* is a prime ideal of $R[X]$ and when I^* is a maximal ideal of $R[X]$. Justify your answers.

2. (a) Show that if a commutative ring R satisfies DCC on ideals (i.e., R is Artinian), then R has only a finite number of maximal ideals.

(b) Give an example to show that (a) may be false if DCC is replaced by ACC (i.e., if R is Noetherian).

III. 1. Let $f: M \to M$ be an R-module homomorphism with $f \circ f = f$. Show that the following statements are equivalent.

(a) f is injective.

(b) f is surjective.

(c) $f = 1_M$.

2. (a) Let G and H be finitely generated abelian groups such that $\mathbb{Z}_n \otimes G \cong \mathbb{Z}_n \otimes H$ for every integer $n \geq 2$. Show that $G \cong H$.

(b) Give an example to show that (a) may be false if G and H are not both finitely generated.

IV. 1. Let F be a subfield of a field L. Show that L/F is an algebraic extension if and only if every subring R of L containing F is a field.

2. Compute the Galois group of $f(X) = X^4 + X + 1 \in \mathbb{Z}_2[X]$.

ALGEBRA PRELIMINARY EXAMINATION
Fall 2011

Attempt all four parts. Justify your answers.

Part I.

1. How many Sylow 2-subgroups does S_5 (the group of permutations of $\{1, 2, 3, 4, 5\}$) have ?
2. Let G be a group of order 231. Show that G is Abelian if and only if G has an Abelian subgroup of order 21.

Part II.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$, *subrings are assumed to contain* 1 *and ring-homomorphisms are assumed to map* 1 *to* 1.

1. Let R be a UFD such that each maximal ideal of R is a principal ideal. Prove that R is a PID.
2. Let $\mathbb{R}[[X]]$ denote the power-series ring in a single indeterminate X over the field of real numbers $\mathbb{R}$. If T is a multiplicative subset of $\mathbb{R}[[X]]$ containing 1 but not containing 0, then show that either $T^{-1}\mathbb{R}[[X]] = \mathbb{R}[[X]]$ or $T^{-1}\mathbb{R}[[X]]$ is a field.

Part III.

Note : *Rings are assumed to be commutative and with* $1 \neq 0$ *and modules are assumed to be unitary.*

1. Let R be an integral domain and I an ideal of R. Show that there exists a surjective R-module homomorphism $f : I \to R$ if and only if I is a nonzero principal ideal.
2. Let K be a field, X an indeterminate over K and M a finitely generated $K[X]$-module. Show that M is a projective $K[X]$-module if and only if M is $K[X]$-module isomorphic to $K[X] \otimes_K V$ for some finite dimensional K-vector space V.

Part IV.

1. Let K be a field and F a subfield of K. The group of units of K is denoted by $K^\times$. Suppose $f \in F[X]$ is a monic irreducible polynomial and $a, b \in K^\times$ are such that $f(a) = 0 = f(b)$. Show that the subgroup of $K^\times$ generated by a, is isomorphic to the subgroup of $K^\times$ generated by b.
2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 4 such that the Galois group of f (over $\mathbb{Q}$) is a group of order 6. Show that f has a root in $\mathbb{Q}$.

Algebra Preliminary Exam **August 2010**

Attempt all problems and justify all answers. All rings have an identity $1 \neq 0$, ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let $f : G \to H$ be a surjective homomorphism of finite groups and $y \in H$ with $|y| = n$. Show that there is an $x \in G$ with $|x| = n$.

2. Let p and q be primes, $p \geq q$, $n \geq 1$, and G a group with $|G| = p^n q$. Show that G has a normal subgroup H of order p^n. (Hint: do the $p > q$ and $p = q$ cases separately.)

II. 1. Let R be a commutative ring with distinct prime ideals P and Q with $P \cap Q = \{0\}$. Show that R is isomorphic to a subring of the direct product of two fields.

2. Let p and q be positive primes. Show that the polynomial $f(X) = X^3 + pX^2 + q \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Q}[X]$.

III. 1. Let A and B be finite abelian groups with $|A| = m$ and $|B| = n$. Show that $\mathrm{Hom}_{\mathbb{Z}}(A,B) = 0$ if and only if $\gcd(m,n) = 1$.

2. Let A be a submodule of a projective R-module B. Show that A is projective if B/A is projective.

IV. 1. Let $K \subseteq F$ and $K \subseteq L$ be subfields of a field M with $[F:K] = p$ and $[L:K] = q$ for distinct primes p and q. Show that $F \cap L = K$, and that $F = K(\alpha)$ and $L = K(\beta)$ for any $\alpha \in F - K$ and $\beta \in L - K$.

2. Let K be a field and $f(X) \in K[X]$ be irreducible and separable with $\deg(f(X)) = n$. Show that if the Galois group G of $f(X)$ over K is abelian, then $|G| = n$.