Math Ecology Prelim Exam, August 29, 2024

Show all of your work. Justify your answers.

Print Name _____

1. Solve this optimal control problem with a stock that can be reinvested to expand capacity or sold for revenue. Find the optimal control and state and corresponding adjoint function.

$$\max \int_0^T (1-u)x \, dt$$

$$x' = ux \qquad x(0) = x_0 > 0$$

   The control $u(t)$ represents the fraction of stock to be reinvested (per time).

2. Determine the asymptotic sex ratio for the female-male model with F as the number of females and M as the number of males:

   $\frac{dF}{dt} = -\mu_f F + b_f \sqrt{FM}$

   $\frac{dM}{dt} = -\mu_m M + b_m \sqrt{FM}$

   Consider changing variables to $R^2(t) = F(t)$ and $S^2(t) = M(t)$. State any needed assumptions. Note that μ_f, μ_m, b_f, b_m are positive constants.

3. Consider the following delayed logistic equation in which the delay T appears in the intrinsic growth rate.

$$x_{t+1} = r x_{t-T}(1 - x_t), \ r > 0.$$

   (a) Compute the equilibria and determine the linearization around the non-zero equilibrium.

   (b) Derive a characteristic equation for the non-zero equilibrium, expressed as $r = g(\lambda)$ for some function g.

   (c) For delay $T = 1$, determine under what conditions the nonzero equilibrium is unstable.

4. Consider this nonlinear PDE for a population with $x \in \mathbb{R}$ and $t \geq 0$:

$$u_t = D u_{xx} + f(u)$$

   with D positive. Derive the necessary conditions and state the assumptions for the general function f so that a traveling wave solution connecting $(1,0)$ and $(0,0)$ exists. Find the minimal wave speed.

5. A Leslie matrix for an age-structure model is given by

$$
L \;=\; \begin{bmatrix} 0 & \frac{3a^2}{2} & \frac{3a^3}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}
$$

with $a > 0$.

a. Interpret the terms of the matrix.

b. Find the eigenvalues and determine the dominant eigenvalue and its eigenvector.

c. Determine the stable age distribution.

d. Show that L^k is a positive matrix for some k.

6. Consider the following model with one space dimension for a coastal population. Derive the general solution, and explain any results about minimal patch size for this model.

$\frac{\partial n}{\partial t} = rn + D\frac{\partial^2 n}{\partial x^2}$ on $0 < x < L$ and $0 < t$.

$n(0, t) = 0$ for $t > 0$

$\frac{\partial n}{\partial x}(L, t) = 0$ for $t > 0$

$n(x, 0) = f(x)$ for $0 < x < L$.

7. Consider a birth and death process for a population whose fecundity increases with population size. We will model only females and assume that males are non-limiting. Assume that this process has the following attributes:

- All individuals act independently of each other and the probability of two or more births/deaths happening in a small time interval $(t, t + \Delta t]$ is $o(\Delta t)$.

- The probability of any given female dying within a time period $(t, t + \Delta t]$ is $\mu \Delta t + o(\Delta t)$.

- The probability of any given female giving birth to one new female within a time period $(t, t + \Delta t]$ is similar, $\beta \Delta t + o(\Delta t)$, but the birth rate β is a function of the current female population. If there is one female, the birth rate is β_0. If there is more than one female, we add a small $\gamma > 0$ to β for each additional female. In other words, for n females, $\beta = \beta_0 + \gamma(n - 1)$.

(a) Using transition probabilities for this process, derive a system of differential equations that describes how the probability of having n females, $p_n(t)$, evolves with time. Include an initial condition that assumes we start with n_0 females at time $t = 0$.

(b) Derive a PDE for the probability generating function along with appropriate initial condition.

8. Construct a Lotka-Volterra predator-prey model with the same harvest rate on both the predator and prey. Then answer the following:

a. Nondimensionalize the model.

b. Find all equilibrium points and determine their stability.

c. According to your results, what happens to the relative abundance of predators and prey as harvest decreases? Explain why such a result is counterintuitive.

Math Ecology Preliminary Exam, January 2024

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

Each problem is worth 10 points.

1. The Leslie matrix for an insect population is

$$\begin{bmatrix} 0 & 20 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}.$$

   The age classes are 0 - 1 weeks, 2-3 weeks, and 4-5 weeks. Notice that the time step here is 2 weeks.

   (a) Explain the meaning of each element in this matrix.

   (b) Find the long term growth rate of the population. Is the population growing or declining?

   (c) Find the long term population structure. What proportion of the population is between 2 and 3 weeks old when the population is at an equilibrium proportion structure?

2. Explain each term in the PDE model below. Give an example of a possible biological population that might be represented this way.

$$u_t = u_{xx} + u_{yy} + bu_x + ru(1 - \frac{u}{K}), \text{ for } x, y \in (0,1) \times (0,1) \text{ and } 0 < t$$

$$u(x,y,0) = f(x,y) \text{ on } (0,1) \times (0,1) \text{ with } 0 = t$$
$$u(0,y,t) = 0 \text{ on } \{0\} \times (0,1) \text{ with } 0 < t$$
$$u(x,0,t) = 0 \text{ on } (0,1) \times \{0\} \text{ with } 0 < t$$
$$u(1,y,t) = 0 \text{ on } \{1\} \times (0,1) \text{ with } 0 < t$$
$$u(x,1,t) = 0 \text{ on } (0,1) \times \{1\} \text{ with } 0 < t$$

Assume b, r, and K are positive constants, and the function f is non-negative.

3. Consider a different version of the lagged logistic equation difference equation (with delay $T = 1$) than the one we discussed in class:

$$N_{t+1} = \left[1 + r\left(1 - \frac{N_t}{K}\right)\right]N_{t-1}.$$

Assume $r, K > 0$. Explain what this equation models compared with the situation where the lag is in the carrying capacity term instead. For the above equation, describe the local behavior of solutions nearby each equilibrium including information about stability, oscillations, and bifurcations (if any). What does this tell you about populations that exhibit behavior as modeled by this equation?

4. For the discrete-time Nicholson-Bailey model of host-parasitoid interactions, something similar to a linear birth process can be used to derive the per-host probability of parasitization in a given time step. Let $p_n(t)$ be the probability that a given host has been found by a parasitoid n times by time t, and assume that the probability of an encounter in some small Δt is equal to $\mu \Delta t$ where μ is constant.

   (a) Assuming that $n = 0$ at time $t = 0$, derive a system of ODEs for $\frac{dp_n}{dt}$ and solve for the probabilities $p_n(t)$.

   (b) State the probability that the host escapes being parasitized at time $t = 1$ as a function of μ.

5. The following system has a (real number) as a parameter.

$$\frac{dx}{dt} = -y + ax(x^2 + y^2)$$

$$\frac{dy}{dt} = x + ay(x^2 + y^2)$$

Investigate the stability and trajectory behavior of this system near (0,0).

6. Write down a linear age-structured model for the number of female births of a fictitious zoo animal as a function of time assuming the following demographics:

   • The zoo starts with three age-18 females, as well as several males.

   • These animals reach sexual maturity at age 6 and are reproductive until age 20.

   • The year-to-year survival probability is independent of age and equal to 0.97.

- Females give birth to, on average, 0.04 females per year before the age of 15 and 0.02 females per year from the age of 15 to 20.

7. Derive the formula for the solution to the problem below using the heat kernel representation and a reflection of the function f. Verify that the boundary condition at $x = 0$ is satisfied.

$$u_t = u_{xx} \text{ on } 0 < x \text{ and } 0 < t$$
$$u(x, 0) = f(x) \text{ for } 0 < x$$
$$u_x(0, t) = 0 \text{ for } 0 < t.$$

8. Consider the optimal control problem when the control is increasing the population,

$$\max_u x(4) - \int_0^4 u^2(t)dt$$

$$\text{subject to } x'(t) = x(t) + u(t), \ x(0) = 0,$$
$$0 \le u(t) \le 5.$$

Determine the optimal control, state, and adjoint.

Math Ecology Prelim Exam, August 2023

Show all of your work. Justify your answers.

Print Name _____

1. Consider the following S, I, R epidemic model using the number of susceptible, infected, and recovered individuals respectively.

$$S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + b(I_t + R_t)$$

$$I_{t+1} = I_t(1 - \gamma - b) + \frac{\beta}{N} I_t S$$

$$R_{t+1} = R_t(1 - b) + \gamma I_t.$$

   where $N = S_0 + I_0 + R_0$ with S_0, I_0, R_0 being positive and $0 < \gamma$ and $0 < b$ and

   $$0 < \beta < 1$$

   $$0 < b + \gamma < 1.$$

   (a) Give the interpretation of this model by explaining each term.

   (b) Show that the solutions are non-negative for all time.

   (c) Give conditions that this system has exactly two equilibria and find those two equilibria.

   (d) Discuss the stability of the equilibrium with $I^* = 0$.

2. For this linear system, discuss its stability and how a Hopf bifurcation occurs as r is varied.

$$\frac{dx}{dt} = rx - y$$

$$\frac{dy}{dt} = x + ry$$

3. Describe the phenomenon known as a "homoclinic bifurcation" (also known as a "saddle separatrix loop bifurcation" or an "Andronov-Leontovich bifurcation") in detail. Why is this classified as a global bifurcation and not a local bifurcation?

4. Consider the Volterra integrodifferential equation with exponential distribution kernel,

$$\frac{dN}{dt} = rN(t) \int_0^\infty k(\tau) \left[1 - \frac{N(t - \tau)}{K}\right] d\tau$$

$$N(t) = N_0(t), \quad -\infty < t \leq 0$$

   with

   $$k(\tau) = \alpha e^{-\alpha \tau}$$

(a) Assuming $N(t)$ is a population at time t, and K is a carrying capacity, describe in detail what this equation is modeling including the significance of the parameter α.

(b) Prove that the carrying capacity equilibrium is asymptotically stable for all $r, \alpha > 0$.

5. Suppose we have a discrete branching process in which the number of offspring sired by each individual has a fixed distribution with probability generating function

$$F(x) = \sum_{n=0}^{\infty} p_n x^n$$

where $p_n = P(N_1 = n | N_0 = 1)$. Assume that this distribution has a known mean $R_0 = F'(1) = E(N_1 | N_0 = 1)$ and a known variance $\text{Var}(N_1 | N_0 = 1) = \sigma^2 = F''(1) + R_0 - R_0^2$.

However, instead of starting with a single individual, consider the case where the initial number of individuals N_0 is a random variable with probability generating function $F_0(x)$. Assume that $E(N_0) = \alpha$ and $\text{Var}(N_0) = \beta^2$.

(a) Derive an expression for $E(N_t)$ using the fact that $E(N_t) = F_t'(1)$.

(b) Recall that the variance of N_t can be found via the equation

$$\text{Var}(N_t) = F_t''(1) + F_t'(1) - (F_t'(1))^2.$$

Derive the following recurrence relation for the first term on the right hand side of this equation,

$$F_{t+1}''(1) = R_0 F_t''(1) + \alpha^2 F''(1) R_0^{2t}.$$

(c) Use the substitution $F_t''(1) = R_0^t u_t$ to show that this nonautonomous difference equation has solution

$$F_t''(1) = \begin{cases} F_0''(1) R_0^t + \frac{\alpha^2 (R_0^t - 1) R_0^t F'''(1)}{R_0 (R_0 - 1)}, & R_0 \neq 1 \\ F_0''(1) + \alpha^2 F''(1) t, & R_0 = 1. \end{cases}$$

Recall that the geometric series $\sum_{i=0}^{t-1} x^i = \dfrac{1 - x^t}{1 - x}$ when $x \neq 1$.

(d) Solve for $\text{Var}(N_t)$ in terms of α, β, R_0, and σ^2. Simplify your solution as much as possible.

6. Solve the optimal control problem (that is, find $u^*(t)$, $x^*(t)$ and the corresponding adjoint function) below

$$\max_u \int_0^2 (2x(t) - 3u(t)) \, dt$$

2

subject to

$$x'(t) = x(t) + u(t), \quad x(0) = 5$$

$$0 \le u(t) \le 2.$$

7. Consider the Fisher partial differential equation

$$\frac{\partial n}{\partial T} = rn(1 - n/K) + D\frac{\partial^2 n}{\partial X^2}.$$

Determine the minimum wave speed of a traveling wave solution for this PDE.

8. Nisbet and Gurney (1983) described a general equation for age and mass-dependent population dynamics:

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial a} - \frac{\partial}{\partial m}[gf] - \delta f,$$

where $f(a, m, t)$ represents the density of individuals of age a and mass m at time t, $g = g(a, m, t)$ represents the rate of growth of an individual of mass m and age a at time t. Similarly, $\delta = \delta(a, m, t)$ is the per capita mortality rate.

(a) Under the simplifying assumption that g is independent of mass, age, and time, $\delta = \delta(m, t)$ is age-independent, and that all individuals are born with the same mass $m = m_1$, show that the balance equation simplifies to

$$\frac{\partial \rho(m, t)}{\partial t} = -g\frac{\partial \rho(m, t)}{\partial m} - \delta(m, t)\rho,$$

where $\rho(m, t) = \int_0^\infty f(a, m, t)da$.

(b) Assume the initial mass distribution is $\rho(m, 0) = \rho_0(m)$ and we have boundary condition $\rho(m_1, t) = B(t)$, where $B(t)$ is a known function. Using a change of variables from (m, t) to (μ, τ), show that the characteristic lines are given by the equation $m = gt + \mu$.

(c) Derive the solution to the mass-structured PDE. You may find it helpful to sketch the characteristic lines and carefully consider the lower bound for τ.

3

Math Ecology Preliminary Exam, January 2023

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers. Each problem is worth 10 points.

1. Interpret the terms in this optimal control problem where x represents a population (state) and u the control. Then solve this problem giving the optimal control, state, and adjoint functions.

$$\max_u [x(4) - \int_0^4 u(t)^2 dt]$$

with $x' = x + u$ with $x(0) = 15$ and $0 \le u(t) \le 5$.

2. Derive the formulas for the solutions $u(x,t), v(x,t)$ to the problems below using the heat kernel representation and extensions of the function f. Explain the biological meaning of the boundary conditions and compare solutions. Verify that the boundary condition at $x = 0$ is satisfied in each case. Tell the conditions that the reflections must satisfy.

$$u_t = u_{xx} \text{ on } 0 < x \text{ and } 0 < t$$
$$u(x,0) = f(x) \text{ for } 0 < x$$
$$u_x(0,t) = 0 \text{ for } 0 < t$$

$$v_t = v_{xx} \text{ on } 0 < x \text{ and } 0 < t$$
$$v(x,0) = f(x) \text{ for } 0 < x$$
$$v_x(0,t) = v(0,t) \text{ for } 0 < t$$

1

3. Given the following Leslie matrix for a population model: $\begin{bmatrix} 0 & 9 & 12 \\ 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$

   (a) Find the long term growth rate.

   (b) Find the stable age distribution.

   (c) After a long time, what percentage of the population can be harvested each time period and the total population size will stay the same?

4. For this population PDE

$$\frac{\partial n}{\partial t} = f(n) + \frac{\partial^2 n}{\partial x^2},$$

   with $f(n) = n$ for $0 \le n \le \frac{1}{2}$ and $f(n) = 1 - n$ for $\frac{1}{2} \le n \le 1$,

   shift to traveling wave coordinates. Solve the corresponding differential equation to determine the shape of the wave, when the speed of the wave is $c = 2$. Assume $n(-\infty) = 1$ and $n(+\infty) = 0$.

5. A fish population with harvest is growing according to this differential equation with parameters r, K_0, K, q, E being positive and $K_0 < K$

$$\frac{dx}{dt} = rx(\frac{x}{K_0} - 1)(1 - \frac{x}{K}) - qEx.$$

   (a) What does the threshold K_0 represent?

   (b) Investigate the equilibrium behavior of this model as E varies.

   (c) What assumption is needed on E to make this harvest sustainable?

6. This system represents the interactions of two populations.

$x' = -\alpha x + \beta xy$

$y' = -\gamma y + \delta xy$

   (a) Assuming the parameters are positive, interpret the terms in this equation.

   (b) Analyze the stability behavior of this model and sketch the phase plane.

   (c) Interpret the behavior from a biological viewpoint.

7. A population of B ducks lives on two ponds: one large, one small. Let $N(t)$ be the number of birds on the small pond. You may assume that there are $B - N(t)$ birds on the large pond. Let the probability of a departure from the small pond be given by

$$P[N(t + \Delta t) = n - 1 | N(t) = n] = r_d n \Delta t + o(\Delta t),$$

where r_d is the departure rate. Similarly assume that the probability of an arrival onto the small pond is

$$P[N(t + \Delta t) = n + 1 | N(t) = n] = r_a (B - n) \Delta t + o(\Delta t),$$

where r_a is the arrival rate.

   (a) Derive a system of differential equations for p_n, the probability of n birds on the small pond.

   (b) Derive a partial differential equation for the probability generating function.

8. Assume that a population experiences a uniform death rate of 1.5% at all ages so that the probability of surviving until age a, $l(a)$, is given by $\frac{dl}{da} = -0.015l$. Further assume that for females, menarche is at age 12 and menopause is at age 50 with given maternity function $m(a)$.

   (a) Write down a McKendrick-von Foerster PDE for the population of females aged a at time t, $n(a, t)$, including a boundary condition. Assume the initial age distribution of females is $n(a, 0) = n_0(a)$.

   (b) Assume there exists a solution of the form $n(a, t) = n^*(a)e^{rt}$. Interpret the biological meaning of $n^*(a)$ and r in this solution.

   (c) Using your boundary condition, derive an Euler-Lotka equation for r in terms of $m(a)$.

Math Ecology Preliminary Exam, AUGUST 2022

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

Each problem is worth 10 points.

1. Derive the formula for the solution to the two problems below using the heat kernel representation and a reflection of the function f. Verify that the boundary condition at $x = 0$ in each problem is satisfied.

$$u_t = u_{xx} \text{ on } 0 < x \text{ and } 0 < t$$
$$u(x, 0) = f(x) \text{ for } 0 < x$$
$$u(0, t) = 0 \text{ for } 0 < t$$

$$v_t = v_{xx} \text{ on } 0 < x \text{ and } 0 < t$$
$$v(x, 0) = f(x) \text{ for } 0 < x$$
$$v_x(0, t) = 0 \text{ for } 0 < t$$

If the initial population function was the same for these two problems, which resulting population would be the largest?

2. In many bird species, the fecundity and survival of adults is independent of the age of the adult bird. Assume we can represent the population with two classes, juveniles and adults.

a) Set up a matrix population model to describe the female population dynamics, assuming that the juveniles do not reproduce and that the juvenile stage lasts only one year. Adult females may live more than one year.

b) Find the eigenvalues and eigenvectors of the population matrix for this model.

c) Find the dominant eigenvalue, the growth rate, and the long term behavior. Determine the long term fraction of the population that are female juveniles.

3. Consider the following predator-prey model that has been proposed for herbivore-plankton population dynamics. Suppose that all parameters are positive, and $K > C$.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{BP}{C+P}H$$

$$\frac{dH}{dt} = D\frac{P}{C+P}H - AH^2$$

(a) Briefly describe the biological meaning of each term in the equations above.

(b) Determine a change of variables so that the equation above can be expressed in the dimensionless form

$$\frac{dx}{d\tau} = x\left[(1 - x/k) - \frac{y}{1+x}\right]$$

$$\frac{dy}{d\tau} = dy\left(\frac{x}{1+x} - ay\right)$$

(c) Neatly sketch the nullclines and the directions of the tangent vectors to the solution curves along each nullcline. Clearly mark the equilibria. NOTE: you do NOT need to explicitly compute the equilibria.

(d) Using your response to part (c), describe the possible types of phase portraits near the interior equilibrium point (x^*, y^*), where x^*, $y^* > 0$.

4. Suppose that each parent has exactly three children. Each child has a probability $m = 0.5$ of being male and a probability $f = 0.5$ of being female. Assume that the number of male offspring can be modeled as a branching process, beginning with one male $(N_0 = 1)$.

(a) Write down the probability generating function for the number of male offspring after one generation. Use this generating function to compute the expected number of male offspring after one generation.

(b) Calculate the extinction probability for this male lineage.

5. Consider a birth-death-emigration process described by the following probabilities

$$P\{\Delta N = +1 | N(t) = n\} = \beta n \Delta t + o(\Delta t)$$
$$P\{\Delta N = -1 | N(t) = n\} = (\mu n + \nu)\Delta t + o(\Delta t)$$

(a) Derive differential equations for the probability mass functions $p_n(t)$.

(b) Derive a partial differential equation for the probability generating function $F(t, x)$.

(c) Assume that $\beta > \mu$. What is the probability that the population is of size zero at equilibrium?

6. In the past, it has been notoriously difficult to breed giant pandas in captivity. There is a discrete mating season which lasts for a very short time per female and occurs only once a year. Suppose that a zoo starts with three females aged 10, 15, and 18 as well as several males. Assume that giant pandas reach sexual maturity at age 6 and are reproductive until age 20. For simplicity, assume that there is an age independent mortality rate of 0.03 and that female giant pandas give birth, on average, to 0.03 females before the age of 15 and 0.02 females from the age of 15 to 20.

(a) Write down a linear, age-structured model for giant panda births as a function of time.

(b) Write down the Euler-Lotka equation for this model and show that that equation has exactly one positive root.

7. Derive the solution of this size-structured population PDE.

$$\frac{\partial n(s, t)}{\partial t} + 0.5\frac{\partial n(s, t)}{\partial s} = -\mu(2s, t)n(s, t) \text{ on } 0 < s \text{ and } 0 < t$$

with $\mu(2s, t)$ as a non-negative per capita mortality rate.

Suppose $n(0, t) = B(t)$, a given known function.

The initial size distribution is $n(s, 0) = n_0(s)$.

8. Explain why spatial features may be included in some models. Summarize alternative ways that ecologists account for spatial aspects of natural systems using models.

Math Ecology Preliminary Exam, AUGUST 2021

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers. Each question is worth 10 points.

1. Consider the following delayed logistic equation in which the delay T appears in the intrinsic growth rate.

$$x_{t+1} = rx_{t-T}(1 - x_t), \ r > 0.$$

   (a) Compute the equilibria and determine the linearization around the non-zero equilibrium.

   (b) Derive a characteristic equation for the non-zero equilibrium, expressed as $r = g(\lambda)$ for some function g.

   (c) For delay $T = 1$, determine under what conditions the nonzero equilibrium is unstable.

2. Consider the following 1D spatial model for a coastal population with emigration:

$$\frac{\partial n}{\partial t} = rn + D\frac{\partial^2 n}{\partial x^2}$$

$$n(0, t) = 0$$

$$\frac{\partial n}{\partial x}(L, t) = 0$$

$$n(x, 0) = n_0(x).$$

   Find the general solution to this problem and solve for the critical patch size.

3. Consider a theoretical population in which only half of all individuals born survive to age 1, but then there is negligible mortality afterward. Females age 1 do not give birth, but on average, age 2 females give birth to 4 other females and age 3 females give birth to 2 other females. Older females do not give birth.

   (a) Using a difference equation approach, determine if the number of births will grow or decay over the long run for an arbitrary, nontrivial population of founder females.

   (b) Assuming that the dominant eigenvalue in part (a) is a positive, real number less than 2, use sensitivity analysis to determine which of the three parameters (survivorship to age 1+, maternity rate at age 2, or maternity rate at age 3) has the greatest effect on the asymptotic, geometric growth rate of the population.

1

4. Suppose the distribution of male offspring in a population is given by the following probabilities

$$p_0 = 1/2$$

$$p_n = \frac{1}{5}\left(\frac{3}{5}\right)^{n-1}, \quad \text{for } n = 1, 2, \ldots.$$

(a) Derive the probability generating function for the number of male offspring from one male.

(b) Compute the asymptotic extinction probability.

5. Suppose an industry regularly inputs waste in an area and the differential equation of the contaminant level is given by the following differential equation, with control u.
$x'(t) = x(t) + 1 - u(t)x(t)$ with $0 \le u(t) \le 3$ and $x(0) = 0$.

Solve this optimal control problem with $T > 2\ln(2)$:

$$\min_u \int_0^T [x(t) + u(t)]dt$$

6. Consider this model of rabies in foxes with positive constants r, D, a.

$$\frac{\partial S}{\partial t} = -rSI$$

$$\frac{\partial I}{\partial t} = rSI - aI + D\frac{\partial^2 I}{\partial x^2}$$

(a) Change variables to nondimensionalize this system.

(b) Using a traveling wave solution on your system from part (a) with $z = x - ct$ and positive speed c, and with
$S(\infty) = 1$ and $I(\infty) = 0$
$S'(\infty) = 0$ and $I(-\infty) = 0$,
determine the fraction of the original susceptible population that survives the epizoonotic wave.

2

(c) Find the minimum wave speed and give the corresponding needed assumptions.

7. A fish population, which would otherwise undergo logistic growth, is being harvested. The number of fish, N, can be described by the model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{HN}{A + N},$$

where $r > 0$, $K > 0$, $A > 0$, and $H \geq 0$.

(a) Describe how the harvesting term, $\frac{HN}{A+N}$, depends on the population size N. Give a biological interpretation of the parameter A.

(b) A change of variables leads to the following nondimensional form

$$\frac{dx}{d\tau} = x(1 - x) - \frac{hx}{a + x}.$$

Investigate equilibria depending on the values of a and h and determine the stability of those equilibria.

(c) Sketch the boundaries between regions in the (a, h)-plane for which there are different numbers of equilibria, or different stability of equilibria.

8. Construct a predator-prey model to answer the following question: How can a decrease in harvesting lead to an increase in the relative abundance of predators but a decrease in the relative abundance of prey? Explain why such a result is counter-intuitive, carry out analysis of your model, use this to justify how the model you constructed assists in analyzing the question.

Math Ecology Preliminary Exam, AUGUST 2020

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

Problems 1 - 7 are worth 10 points each, and problems 8 and 9 are worth 5 points each.

1. Describe what types of density dependence features might occur in population models. What are the biological or mathematical reasons why these features would possibly be included? Give examples of such models which feature some aspect of density dependence. In these examples, include at least one example in which density dependence would enhance the population and one in which density dependence lowers the population size.

2. Consider this two-sex model for females F and males M:

$$\frac{dF}{dt} = -\mu_f F + b_f \Lambda(F, M)$$

$$\frac{dM}{dt} = -\mu_m M + b_m \Lambda(F, M),$$

with positive constants μ_f, μ_g, b_f, b_m.

a. Suppose the reproduction function $\Lambda(F, M) = F$. Explain a biological situation for which this choice may be reasonable. Solve for the sex ratio as $t \to \infty$. Give any needed assumptions.

b. Suppose the reproduction function $\Lambda(F, M) = FM$, $M(0) = F(0)$, and $\mu_f = \mu_g$, $b_f = b_m$. Describe the stability of the resulting populations. Give any needed assumptions.

3. Suppose for a population of females, the number of females born on average to a female of age a is
$m_a = m$ for $a \geq 3$ (with m being a known positive constant) and $m_a = 0$ for $a < 3$.

The fraction of females surviving from birth to age a is l_a with l_3 being a known positive constant and
$P = \frac{l_{a+1}}{l_a}$ for $a \geq 3$.

a. Find the corresponding Euler-Lotka equation.

b. Assume the dominant root is λ_0, find the sensitivity of λ_0 with respect to P in terms of λ_0, l_3, m and P.

4. Derive the explicit solution of this age-structured PDE with $0 < a < \infty$ and $t > 0$.

$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a,t)n(a,t)$

with $\mu(a,t)$ as a non-negative per capita mortality rate.

Suppose $n(0,t) = \int_0^\omega n(a,t)m(a,t)da$ with m as a non-negative maternity function. The initial age distribution is $n(a,0) = n_0(a)$.

5. Let $p(x,t)$ denote the proportional density of the population carrying allele A located at location x and time t and satisfies

$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} + rp(x,t)(1 - p(x,t))$

with $-\infty < x < \infty$ and $t > 0$ and with positive constants D, r. Derive the minimum wave speed for a traveling wave solution with $z = x - ct$ with $p(z) \to 1$ as $z \to -\infty$ and $p(z) \to 0$ as $z \to \infty$.

6. Consider a branching process for a species that undergoes alternating "good" and "bad" years for reproduction. In a good year, an individual has the following probability mass function for the number of offspring:

$$p_{g,0} = 0.2, \qquad p_{g,1} = 0.2, \qquad p_{g,2} = 0.4, \qquad p_{g,3} = 0.2$$

and in a bad year, the probability mass function is:

$$p_{b,0} = 0.5, \qquad p_{b,1} = 0.3, \qquad p_{b,2} = 0.1, \qquad p_{b,3} = 0.1.$$

Assume the population starts on a good year. Derive an expression for the expected growth rate of this population after t years. Is the population expected to grow or decay?

7. Solve this optimal control problem for a fish harvest model.

$$\max_E \left[\frac{N(T)}{2} + \int_0^T E(t)N(t)dt \right]$$

with any control E satisfying $0 \le E(t) \le 1$,

$$N' = N - EN$$

and $N(0) = 1000$. Solve for the optimal control, state, and adjoint explicitly.

8. For the following discrete-time model for spread of a disease, describe the key assumptions and interpret each of the terms on the right hand side of each equation. In the model, B is the number in the newborn class, S_A the number of susceptible adults, I_A number of infected adults and R the number of recovered adults. Assume the rates satisfy, $0 < d_B < 1$, $0 < d_A < 1$, $0 < \gamma < 1$ and $0 < \mu_B < 1$ and the other parameters are positive. Also $A(t) = S_A(t) + I_A(t) + R(t)$.

$$B(t+1) = B(t)(1 - d_B)(1 - \mu_B) + \frac{b_A K}{K + A(t)} A(t)$$

$$S_A(t+1) = [S_A(t)(1 - d_A) + B(t)(1 - d_B)\mu_B] \exp\left(- \beta_A I_A(t)\right)$$

$$I_A(t+1) = I_A(t)(1 - d_A)(1 - \gamma) + [S_A(t)(1 - d_A) + B(t)(1 - d_B)\mu_B]\left[1 - \exp\left(- \beta_A I_A(t)\right)\right]$$

$$R(t+1) = R(t)(1 - d_A) + \gamma I_A(t)(1 - d_A)$$

9. Consider this model with s, b, c positive constants and $0 < a < 1$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} ac & bc \\ s(1-a) & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

a. Determine conditions (if any) for which there are infinitely many equilibrium points.

b. For conditions on the parameters for which a single equilibrium exists, determine conditions on the parameters that ensure that all solutions converge to that equilibrium point as $t \to \infty$.

Math Ecology Preliminary Exam, August 2019

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

1. This model represents two classes in a bee colony. One class is impaired due to a stressor. The operational size of the colony is $N = S + cI$ with c representing the level of behavioral impairment with $(0 \leq c \leq 1)$.

$$\frac{dS}{dt} = bN - \frac{\mu S}{N + \phi} - \beta S$$

$$\frac{dI}{dt} = \beta S - \frac{\mu I}{N + \phi} - \nu I$$

Give the interpretation of the terms of this model. Explain why density dependent terms in this model may act differently than such terms in other population models.

2. The following system has a (real number) as a parameter.

$$\frac{dx}{dt} = -y + ax(x^2 + y^2)$$

$$\frac{dy}{dt} = x + ay(x^2 + y^2)$$

Investigate the stability and trajectory behavior of this system near (0,0).

3. Consider the McKendrick-von Foerster equation

$$\frac{\partial n}{\partial t} + 0.5\frac{\partial n}{\partial s} = -\mu(2s)n - h(N(t))n$$

for a harvested, size structured tree population that grows 0.5 cm per year, along with the conditions

$$n(0, t) = B(t)$$
$$n(s, 0) = n_0(s)$$

a. Derive these equations from first principles. Explain the meaning of each term appearing in the equation.

b. Solve for the size distribution of trees at any time $t > 0$, $n(s, t)$, assuming that $N(t)$ and $B(t)$ are known functions.

1

4. The following is a model for habitat conversion from forests to agriculture and then to degraded land. Let F be the area covered by forest, A the area devoted to agriculture, U the unused land area (degraded), and P the human population. A simple model for habitat conversion is

$$\frac{dF}{dt} = sU - \delta PF$$

$$\frac{dA}{dt} = \delta PF + bU - aA$$

$$\frac{dU}{dt} = aA - (b + s)U$$

$$\frac{dP}{dt} = rP(1 - \frac{h}{A}P)$$

a. Interpret this model including an explanation of every term.

b. Find all equilibria if $P \equiv 0$. Find their stability. Assume all parameters are strictly positive.

5. Consider the following matrix for a discrete model of a savanna tree population with seedling, juvenile, and adult classes and shrinkage (part of the tree may die, while the rest re-sprouts):

$$\begin{bmatrix} 0 & s_1 & b \\ g_1 & 0 & s_2 \\ 0 & g_2 & r \end{bmatrix}$$

Assume that the rates in the matrix are positive.

a. Interpret the terms of this matrix with respect to assumptions about the classes in the model.

b. Explain how the Perron-Frobenius Theorem can be applied to this matrix model. What does the theorem tell you about the long-term behavior of the population?

6. Consider an asexual population undergoing a branching process that is known to have no more than 2 children per parent. The probability of an individual having zero children is known to be 1/6, but the probability of having 1 or 2 children is only known to be nonzero.

a. Let a represent the probability of having 1 child and b represent the probability of having 2 children. Write down the probability generating function for this branching process.

b. Assume that we start with a single individual; derive a condition on a that will ensure the probability of eventual extinction is less than 1. Under this condition, what is the probability that the population goes extinct?

7. Consider the reaction-diffusion model

$$\frac{\partial n}{\partial t} = f(n) + \frac{\partial^2 n}{\partial x^2}$$

with

$$f(n) = \begin{cases} n, & 0 \le n \le 1/2 \\ 1-n, & 1/2 \le n \le 1. \end{cases}$$

Shift to traveling wave coordinates. Solve the corresponding ordinary differential equation to determine the exact shape of the traveling wave that satisfies $n(-\infty) = 1$ and $n(+\infty) = 0$.

8. Summarize the basic types of quantitative models for a single population. For each, note what the basic assumptions are in the formulation, explain real-world situations in which these assumptions might well be violated, and give examples for how the model might be modified to account for their lack of concordance with observed populations. For one of the general types of single-population models you mention, describe in detail a mathematical analysis of its behavior.

Math Ecology Preliminary Exam, January 2018

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

1. The following system represents populations of a prey N and a predator P

$$\frac{dN}{dT} = rN - c(N - S)P$$

$$\frac{dP}{dT} = b(N - S)P - mP$$

In this model, S of the prey may hide in a refuge and avoid the predator. The parameters r, c, b, m are all positive. Describe the units of all model variables and components and make appropriate changes of variables to nondimensionalize the system. Find all equilbrium points of the transformed system and investigate their stability.

2. For this model of a prey population, N, assume that r, K, c, and a are positive parameters and consider the number of predators, P, as a bifurcation parameter:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{cPN}{a + N}.$$

Give a complete stability analysis for this model and construct the corresponding bifurcation diagram. Explicitly find and label any points on the bifurcation diagram where stability changes.

3. Assume the functions f and g have continuous first partial derivatives in an open set containing the point $(\bar{x}, \bar{y})$. Assume that $(\bar{x}, \bar{y})$ is an equilibrium point of the system

$$x_{t+1} = f(x_t, y_t)$$

$$y_{t+1} = g(x_t, y_t).$$

Prove that the equilibrium point $(\bar{x}, \bar{y})$ is locally stable if

$$|Trace(J)| < 1 + det(J) < 2$$

with J denoting the Jacobian matrix evaluated at the equilibrium point.

4. Derive the average velocity of expansion for large time for the PDE model

$$n_t = rn + Dn_{xx}$$

starting with a point source, where $x \in \mathbb{R}$ and $t > 0$. You can use the heat kernel representation but all other details of the derivation need to be shown. Be explicit in showing how to get the representation for a solution with a point source.

1

5. Provide a summary of alternative approaches that may be used in ecological models to incorporate aspects of uncertainty. For each alternative you describe, give a circumstance for which that approach may be appropriate. What different mathematical analyses are required for the approaches you suggest, and how are those different from those used to analyze models that do not include uncertainty?

6. Below is a model for two populations with $a \in (0, A)$ and $t \in (0, T)$. Interpret each term in the model by explaining in words what the term represents. Include an explanation of the given initial conditions and boundary conditions. Assume μ, m, f_1, f_2, h_1 and h_2 are non-negative functions of their arguments and d is a positive parameter.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -(\mu(a) + h_1(a, t) + m(U(t)) + f_1(v))u$$

$$\frac{dv}{dt} = (f_2(U(t)) - d - h_2(t))v$$

with
$u(a, 0) = u_o(a)$, $v(0) = v_o$ for $a \in [0, A]$ and $t = 0$,

$u(0, t) = \int_0^A \beta(s)u(s, t)ds$ for $a = 0$ and $t \in [0, T]$, and

$$U(t) = \int_0^A u(a, t)da.$$

7. Consider the population model

$$\frac{\partial N}{\partial t} = rN + b\frac{\partial N}{\partial x} + D\frac{\partial^2 N}{\partial x^2}$$

given initial distribution $N(x, 0) = N_0(x)$ for $x \in [0, L]$ and $t > 0$ and boundary conditions $N(0, t) = 0$ and $N(L, t) = 0$. Assume r, b are constants and D is a positive constant.

a. Derive the critical patch size for this model. State any needed conditions on the parameters r, b, D.

b. What can be said about the critical patch size if the boundary conditions are changed to Neumann conditions?

8. Using the following transition matrix for a population model,

$$A = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix},$$

analyze the long-term behavior of this population.

Math Ecology Preliminary Exam, August 2017

No calculators, cell phones, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

1. A rabbit population model has the following maternity and survival parameters:
   $m_1 = 2$
   $m_2 = 8$
   $l_1 = 0.5$
   $l_2 = 0.25$
   with m_a, l_a being 0 for $a > 2$.
   Assume an initial female population of one newborn rabbit and one 1-year-old rabbit.

   a. What is the number of births, population size, and age structure for years 2 and 3?

   b. What is the predicted stable age distribution?

2. a. Let f' be continuous on an interval I and $f : I \to I$. Prove that if $1 + f'(x) \neq 0$ for all x in I, then $x_{t+1} = f(x_t)$ has no 2-cycles in I.

   b. For the equation
   $$x_{t+1} = \frac{a x_t}{b + (x_t)^k}$$

   with $x_0 > 0$ and positive parameters a, b, and k, find conditions on the parameters so that the equation has no 2-cycles.

3. In the model below, let x be defined as the density of a species and r as the intrinsic growth rate of that species and let h_L and h_N be two constants. Assume the parameters in this model are positive.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - h_L x$$
$$\tau \frac{dr}{dt} = r_e - r - (\alpha h_N + \beta h_L)$$

   a. Interpret the the biological meaning of each term of this model and give the units for each variable and parameter.

   b. Find all equilibrium points of this model and investigate their stability.

4. For a population, the number of sons that a typical male has in his lifetime closely follows these probabilities:
$$p_o = \frac{1}{2} \text{ and } p_k = \frac{1}{5}\left(\frac{3}{5}\right)^{k-1} \text{ for } k = 1, 2, ...$$

Find the probability that the descendants of such a typical male will continue forever.

5. Consider this nonlinear PDE for a population with $x \in \mathbb{R}$ and $t \geq 0$:

$$u_t = Du_{xx} + f(u).$$

Derive the necessary conditions and state the assumptions for the general function f for a traveling wave solution connecting $(1,0)$ and $(0,0)$ to exist and find the minimal wave speed.

6. There are a small number of basic interactions that arise between two species when considering population-scale interactions. (a) Describe these basic interactions using a mathematical or graphical format; (b) explain how they differ in terms of what the form of the interactions are; (c) describe the kinds of responses you expect to observe in modeling these interactions; and (d) give examples of biological systems in which these interaction types occur. (e) Choose one of the types of interactions and provide some insight based on your knowledge of modeling results regarding what can happen in the system if a third species is added.

7. Construct a predator-prey model to answer the following question: How can a decrease in harvesting lead to an increase in the relative abundance of predators but a decrease in the relative abundance of prey? Explain why such a result is counter-intuitive, carry out analysis of your model, use this to justify how the model you constructed assists in analyzing the question.

8. Consider this optimal control problem with harvest in a population:

$$\max_{E} \int_0^T e^{-\delta t} pqE(t)N(t)dt$$

subject to

$$\frac{dN}{dt} = f(N) - qEN$$

and $N(0) = K > 0$ and $0 \leq E(t) \leq E_M$. Assume f is a differentiable function.

a. State the necessary conditions for this optimal control problem with a general function f.

b. If the singular case occurs, what condition does N^* satisfy in that case with a general function f.

c. With $f(N) = rN(1 - \frac{N}{K})$, explain the ideas behind a fundamental principle of renewable resources that larger discount rates lead to less biological conservation.

2

Math Ecology Preliminary Exam, August 2016

No calculators, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

1. Consider the following matrix for a discrete model of a population with a juvenile class and two distinct adult classes:

$$\begin{bmatrix} 0 & b & 0 \\ s_1 & 0 & s_3 \\ 0 & s_2 & 0 \end{bmatrix}$$

   (a) Assume that the rates in the matrix are positive. Interpret the terms of this matrix with respect to assumptions about the classes in the model.

   (b) Describe the dynamics of the long-term behavior of the population, giving any assumptions on the parameters to justify your results.

2. Consider this model for an insect population:

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \frac{bN^2}{a^2 + N^2}$$

   where r, K, a and b are positive parameters.

   (a) Nondimensionalize the model equation and show that it can be put into this form

$$\frac{du}{d\tau} = cu(1 - \frac{u}{q}) - \frac{u^2}{1 + u^2},$$

   where u and τ are the nondimensionalized forms of N and t, respectively; and c and q are dimensionless parameters representing combinations of the original parameters.

   (b) Using the nondimensionalized model, determine the equilibria. Graphically determine the various cases of number of biologically feasible equilibria that can exist and the corresponding local stability of the equilibria in each case. Partition the $c - q$ parameter space into regions based on the number of equilibria in them and derive a parametric relationship for the boundaries between these regions.

1

3. The probability generating function for a branching process with $N_0 = 1$ is
   $F(x) = ax^2 + bx + c$ where a, b and c are positive and $F(1) = 1$. Assume $F'(1) > 1$.
   Use this information to show the extinction probability is $\frac{c}{a}$.

4. Derive the critical patch size for the given model of a population, $N(x, t)$:

$$\frac{\partial N}{\partial t} = rN + b\frac{\partial N}{\partial x} + D\frac{\partial^2 N}{\partial x^2}$$

given initial distribution $N(x, 0) = N_0(x)$ for x in [0,L] and $t > 0$; and boundary conditions
$N(0, t) = 0$ and $N(L, t) = 0$. Assume r, b and D are constants with D being positive. Give
the needed assumptions on the parameters to obtain the critical patch size.

5. Consider the discrete Nicholson-Bailey host (H)-parasitoid (P) model:

$$H_{n+1} = kH_n e^{-aP_n} \tag{1}$$

$$P_{n+1} = cH_n \left[1 - e^{-aP_n}\right], \tag{2}$$

where k, a, and c are positive parameters.

(a) Determine the co-existence equilibrium of the above system and state condition(s) on
the equilibrium to ensure that it is biologically feasible. Explain why the condition(s)
are necessary.

(b) Determine the stability of the co-existence equilibrium (assuming the conditions you
found in part (a)).

(c) Regardless of your findings in part (b) for the coexistence equilibrium, what other
way could this model exhibit coexistence dynamics?

6. Consider the following delay differential equation model for a laboratory population of blowflies, where τ is a particular time delay:

$$\frac{dN}{dt} = aN(t - \tau)e^{-bN(t-\tau)} - cN(t)$$

where a, b, and c are positive parameters. Assume that $a > c$.

(a) Determine all of the fixed points of the system.

(b) Write down the equation for the linear approximation to $\frac{dN}{dt}$ near the fixed point $N^* = 0$.

(c) Determine the stability of the extinction equilibrium.

7. Solve this optimal control problem for this population model, in which the control is a source term that affects the population. We seek to increase the population while minimizing the cost of the control.

$$\max_{u} \int_0^2 (x(t) - 1.5u(t))dt$$

with any control u satisifying $0 \leq u(t) \leq 2$,

$$x'(t) = x(t) + u(t)$$

and $x(0) = 5$. Solve for the optimal control, state, and adjoint explicitly.

8. Based upon the text, review articles by Alan Hastings read in class, and the class discussions, provide a brief summary of at least three different mathematical and/or computational approaches used to address questions in spatial ecology. For each of these three approaches, compose one paragraph which describes (using equations as appropriate) the key ideas of the approach, some spatial ecology questions the approach has been used to investigate, and some advantages and disadvantages of the approach.

Math Ecology - Prelim Exam, January 2014

No calculators, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

1. Consider the following differential equations model for a predator P and its prey N. In this model, s of the prey may hide in a refuge and avoid predation:

$$\frac{dN}{dt} = rN - c(N - s)P$$

$$\frac{dP}{dt} = b(N - s)P - mP.$$

   (a) Nondimensionalize the above system of equations. The parameters r, c, s, B and m are positive.

   (b) Using the Bendixon-Dulac criterion with function $B(x, y) = \frac{1}{xy}$, determine if the system exhibits any periodic orbits.

2. For this discrete population model with juveniles J and adults A:

$$J_{k+1} = sJ_k + fA_k$$

$$A_{k+1} = rJ_k.$$

   Assume s, f and r are non-negative parameters. Determine the relationship that the parameters must satisfy to achieve a long-term growth rate of exactly 1.

3. Consider the model with three interacting populations:

$$x' = x(1 - x) - xz$$

$$y' = ry(1 - y) - yz$$
$$z' = z(2x + 2y - 1)$$

   (a) Interpret the terms in this model with parameter $r > 0$.

   (b) Find all the biological feasible equilibrium points.

   (c) Examine the stability of those equilibrium points.

4. Consider a birth and death process with constant immigration:

$$P[\Delta N = +1|N(t) = n] = \beta n \Delta t + I \Delta t + o(\Delta t^2).$$
$$P[\Delta N = -1|N(t) = n] = \mu n \Delta t + o(\Delta t^2).$$
$$P[\Delta N = 0|N(t) = n] = 1 - [(\beta + \mu)n + I]\Delta t + o(\Delta t^2).$$

Assume that the mortality rate is greater than the birth rate: $\mu > \beta$.

(a) Derive differential equations for the probabilities.

(b) Derive a partial differential equation for the probability generating function.

(c) What are the odds that the population is of size zero at equilibrium?

5. Derive the critical patch for this population model in terms of the non-negative parameters r, a with $r > \frac{a^2}{4}$.

$$\frac{\partial n}{\partial t} = rn + a\frac{\partial n}{\partial x} + \frac{\partial^2 n}{\partial x^2}$$

for $0 < x < L$ and $t > 0$ and with conditions $n(x,0) = n_0(x)$ non-negative, $n(0,t) = 0$ and $n(L,t) = 0$.

6. Let x be the state variable and $r \in \mathbb{R}$ a parameter in the following differential equation:

$$\frac{dx}{dt} = x + \frac{rx}{1 + x^2}$$

Sketch the bifurcation diagram with respect to r for the above equation. Be sure to show and label the stability of each branch. Identify and label the type(s) of bifurcation(s) in your diagram. Give the bifurcation value(s) of r.

7. Solve this optimal control problem for a fish harvest model.

$$\max_{E} \int_0^T E(t)N(t)dt$$

where T is fixed, with any control E satisifying $0 \le E(t) \le M$,

$$N' = N - EN$$

and $N(0) = 1$. Solve for the optimal control, state, and adjoint explicitly.

8. For this population model with radial diffusion,

$$\frac{\partial n}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial n}{\partial r})$$

with r as the radial coordinate, the solution corresponding to a point release of n_o individuals at $r = 0$ at $t = 0$ is

$$n(r,t) = \frac{n_o}{4\pi Dt}e^{-\frac{r^2}{4Dt}}.$$

Now for this equation,

$$\frac{\partial N}{\partial t} = \alpha N + D\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial N}{\partial r}),$$

estimate the average velocity of spread for large times. Note that α and D are positive constants.

Math Ecology - Prelim Exam, January 2014

No calculators, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

1. Consider the following differential equations model for a predator P and its prey N. In this model, s of the prey may hide in a refuge and avoid predation:

$$\frac{dN}{dt} = rN - c(N - s)P$$

$$\frac{dP}{dt} = b(N - s)P - mP.$$

   (a) Nondimensionalize the above system of equations. The parameters r, c, s, B and m are positive.

   (b) Using the Bendixon-Dulac criterion with function $B(x, y) = \frac{1}{xy}$, determine if the system exhibits any periodic orbits.

2. For this discrete population model with juveniles J and adults A:

$$J_{k+1} = sJ_k + fA_k$$

$$A_{k+1} = rJ_k.$$

   Assume s, f and r are non-negative parameters. Determine the relationship that the parameters must satisfy to achieve a long-term growth rate of exactly 1.

3. Consider the model with three interacting populations:

$$x' = x(1 - x) - xz$$

$$y' = ry(1 - y) - yz$$

$$z' = z(2x + 2y - 1)$$

   (a) Interpret the terms in this model with parameter $r > 0$.

   (b) Find all the biological feasible equilibrium points.

   (c) Examine the stability of those equilibrium points.

4. Consider a birth and death process with constant immigration:

$$P[\Delta N = +1|N(t) = n] = \beta n\Delta t + I\Delta t + o(\Delta t^2).$$
$$P[\Delta N = -1|N(t) = n] = \mu n\Delta t + o(\Delta t^2).$$
$$P[\Delta N = 0|N(t) = n] = 1 - [(\beta + \mu)n + I]\Delta t + o(\Delta t^2).$$

Assume that the mortality rate is greater than the birth rate: $\mu > \beta$.

(a) Derive differential equations for the probabilities.

(b) Derive a partial differential equation for the probability generating function.

(c) What are the odds that the population is of size zero at equilibrium?

5. Derive the critical patch for this population model in terms of the non-negative parameters r, a with $r > \frac{a^2}{4}$.

$$\frac{\partial n}{\partial t} = rn + a\frac{\partial n}{\partial x} + \frac{\partial^2 n}{\partial x^2}$$

for $0 < x < L$ and $t > 0$ and with conditions $n(x, 0) = n_0(x)$ non-negative, $n(0, t) = 0$ and $n(L, t) = 0$.

6. Let x be the state variable and $r \in \mathbb{R}$ a parameter in the following differential equation:

$$\frac{dx}{dt} = x + \frac{rx}{1 + x^2}$$

Sketch the bifurcation diagram with respect to r for the above equation. Be sure to show and label the stability of each branch. Identify and label the type(s) of bifurcation(s) in your diagram. Give the bifurcation value(s) of r.

7. Solve this optimal control problem for a fish harvest model.

$$\max_E \int_0^T E(t)N(t)dt$$

where T is fixed, with any control E satisfying $0 \le E(t) \le M$,

$$N' = N - EN$$

and $N(0) = 1$. Solve for the optimal control, state, and adjoint explicitly.

8. For this population model with radial diffusion,

$$\frac{\partial n}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial n}{\partial r})$$

with r as the radial coordinate, the solution corresponding to a point release of n_o individuals at $r = 0$ at $t = 0$ is

$$n(r, t) = \frac{n_o}{4\pi Dt}e^{-\frac{r^2}{4Dt}}.$$

Now for this equation,

$$\frac{\partial N}{\partial t} = \alpha N + D\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial N}{\partial r}),$$

estimate the average velocity of spread for large times. Note that α and D are positive constants.

Math Ecology Prelim Exam, Fall 2014

No calculators or laptops allowed. Justify your answers.

Print Name _____

1. Consider the system

$$\frac{dX}{dt} = rX - cXY + csY, \tag{1a}$$

$$\frac{dY}{dt} = bXY - mY - bsY. \tag{1b}$$

where r, b, c, m and s are positive parameters.

(a) Nondimensionalize the system.

(b) Determine the equilibria and determine their stability.

(c) Using the Bendixon-Dulac criterion with function $B(x,y) = \frac{1}{xy}$ determine if there exist any periodic orbits.

2. Consider the difference equation

$$x_{n+1} = x_n e^{a(1-x_n)}, \tag{2}$$

with $x_0 > 0$ and $a > 0$.

(a) Determine the fixed points and their stability.

(b) Determine 3 values of a that show different dynamics. Illustrate the dynamics via cobweb plots.

3. Consider a population with exponential growth in which time has been rescaled so that $x'(t) = x(t)$. Let the control u be a harvest term. We seek to minimize the population, assuming that p is the cost of a unit of population, while minimizing the cost implementing this control:

$$\min_{u} \int_0^T [px(t) + u(t)]dt, \tag{3a}$$

subject to

$$x'(t) = x(t) - u(t), \tag{3b}$$

with $x(0) = 10$ and $0 \leq u(t) \leq 1$. Assume T is a positive constant and set $p = 2$.

Find the optimal control, and the corresponding state and adjoint functions.

4. Consider a branching process where the probability of having k offspring is given by p_k, where

$$p_0 = \frac{1}{8}, \quad p_1 = \frac{3}{8}, \quad p_2 = \frac{3}{8}, \quad p_3 = \frac{1}{8}, \text{ and } p_k = 0, \text{ for } k > 3. \tag{4}$$

(a) Compute the corresponding probability generating function and compute the net reproductive rate.

(b) Find the probability that a lineage starting with a single individual goes extinct.

5. Let $N(t)$ denote the density of prey at time t. Let $S(t)$ and $I(t)$ denote the density of susceptible and infectious predators, respectively, at time t.

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) - aN(t)(S(t) + I(t)) \tag{5a}$$

$$\frac{dS}{dt} = (b + \varepsilon aN(t))(S(t) + I(t)) - \beta S(t)I(t) - mS(t) \tag{5b}$$

$$\frac{dI}{dt} = \beta S(t)I(t) - (m + \mu)I(t), \tag{5c}$$

with initial conditions

$$N(0) = N_0, \quad S(0) = S_0, \quad I(0) = I_0, \tag{5d}$$

(a) Give an interpretation of the terms in this model.

(b) Analyze the stability of the nontrivial predator-free equilibrium point. Give any needed assumptions on parameters. Assume that all parameters are positive.

(c) Discuss the possible types of biological feasible equilibrium points for this model. It is not necessary to exactly find the equilibrium points and study their stability.

6. Let $n_{a,t}$ denote the population of age a at time t and consider the following age-structured model:

$$n_{0,t+1} = 104n_{1,t} + 160n_{2,t} \tag{6a}$$

$$n_{1,t+1} = 0.01n_{0,t} \tag{6b}$$

$$n_{2,t+1} = 0.3n_{1,t}, \tag{6c}$$

where $n_{a,0}$ is given for $a = 0, 1, 2$.

Determine if the system has a stable age distribution or not, and if it does, express the stable age distribution in terms of the growth factor for the system.

7. For this equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2},$$ (7a)

with $-\infty < x < \infty$ and $t > 0$, the solution corresponding to a point release of n_0 individuals at the point $x = 0$ at $t = 0$ is given by

$$n(x,t) = \frac{n_0}{2\sqrt{\pi D t}}e^{-\frac{x^2}{4Dt}}.$$ (7b)

Using the above result, for the model

$$\frac{\partial N}{\partial t} = rN - \beta\frac{\partial N}{\partial x} + D\frac{\partial^2 N}{\partial x^2},$$ (7c)

estimate the average velocity of the spread for large times, in the positive x-direction starting from the same point release. Assume r, β and D are positive constants, and $r > \frac{\beta^2}{4D}$.

8. Let $n(a,t)$ denote the population of age a at time t and consider the system:

$$\frac{\partial n}{\partial a} + \frac{\partial n}{\partial t} = -\mu(N)n,$$ (8a)

$$n(0,t) = B(t),$$ (8b)

$$n(a,t) = n_0(a),$$ (8c)

where

$$N(t) = \int_0^\infty n(a,t)\,da,$$ (8d)

$$B(t) = \int_0^\infty m_0 e^{-\gamma a} n(a,t)\,da,$$ (8e)

μ and n_0 are nonnegative functions, and m_0 and γ are positive constants.

(a) Derive a system of initial value problems for $N(t)$ and $B(t)$.

(b) Suppose that the system from part (a) has been solved for $N(t)$ and $B(t)$. Determine the population distribution $n(a,t)$ for all (a,t).

Math Ecology - Prelim Exam, August 2013

No calculators, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

1. Consider this simple model of Celiac disease:

$$g' = -d_1 g,$$

$$z' = \frac{c_1 g^w}{a_1 + g^w} - d_2 z,$$

$$r' = \sigma r(1 - r) - \frac{c_2 r z^j}{a_2 + z^j},$$

where $g(t)$ is the concentration of gluten, $r(t)$ is the electrical resistance (which is measure of permeability of the small intestine) and $z(t)$ is the concentration of inflammatory cytokines.

Assume $w \geq 1$, $j \geq 1$, and all the constants, $d_1, d_2, \sigma, c_1, c_2, a_1,$ and a_2 are positive.

(a) Find the equilibrium points of this system and analyze their stability.

(b) Assume $g(0), z(0),$ and $r(0)$ are positive, and show that $g(t), z(t)$ and $r(t)$ remain positive for all time.

2. Consider this population model.

$$\frac{\partial N}{\partial t} = rN + D \frac{\partial^2 N}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$

with initial distribution $N(x, 0) = N_0(x)$ for $x > 0$, and $N_0(x)$ being bounded and non-negative with $N_0(0) = 0$ and $N_0'(0) = 0$. Assume r and D are positive constants.

(a) Give an explicit representation to the solution of this PDE with the given initial condition and the boundary condition: (A) $N(0, t) = 0$ for $t > 0$.

(b) Give an explicit representation to the solution of this PDE with the given initial condition and the boundary condition: (B) $N_x(0, t) = 0$ for $t > 0$

(c) Verify that the boundary condition holds in each representation. Which solution would be larger? Give a reason from your representations and a reason from a biological viewpoint.

3. For this discrete population model, assume $0 < a < K, r > 0$, and $x_0 > 0$. For $k = 1, 2, \ldots$ assume $0 \le h_k < 1$, and

$$x_{k+1} = [x_k + rx_k (x_k - a)(K - x_k)](1 - h_k).$$

(a) Describe the events and their order of occurence represented in this model.

(b) Write a new version of this model if the events were reordered.

(c) Examine the equilibrium and long term stability of the model stated in this problem, assuming $h_k = 0$ for all k.

4. Consider the reaction-diffusion model

$$\frac{\partial n}{\partial t} = f(n) + \frac{\partial^2 n}{\partial x^2}$$

with

$$f(n) = \begin{cases} 4n, & 0 \le n \le \frac{1}{2} \\ 4(1 - n), & \frac{1}{2} \le n \le 1 \end{cases}$$

Determine the exact shape of the traveling wave solution that satisfies $\lim_{x \to -\infty} n(x, t) = 1$ and $\lim_{x \to \infty} n(x, t) = 0$.

5. The Leslie Matrix for an age-structured population is given by

$$L = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}.$$

(a) Interpret the terms of this matrix with respect to assumptions about birth rates and survival rates for the various age classes.

(b) Derive the Euler-Lotka equation for this system, and calculate the eigenvalues for this system.

(c) Characterize the long-term behavior of the population.

6. Consider a birth and death process with constant immigration:

$$P[\Delta N = +1 | N(t) = n] = \beta n \Delta t + I \Delta t + o(\Delta t^2).$$
$$P[\Delta N = -1 | N(t) = n] = \mu n \Delta t + o(\Delta t^2).$$
$$P[\Delta N = 0 | N(t) = n] = 1 - [(\beta + \mu)n + I]\Delta t + o(\Delta t^2).$$

Assume that the mortality rate is greater than the birth rate: $\mu > \beta$.

(a) Derive differential equations for the probabilities.

(b) Derive a partial differential equation for the probability generating function.

(c) What are the odds that the population is of size zero at equilibrium?

7. A spatially implicit model for the proportion of islands occupied by a particular species is given by:

$$\frac{dp}{dt} = (m + cp)(1 - p) - ep,$$

where $p(t)$ is the fraction of occupied islands at time t, $m \geq 0$, $c > 0$, and $e > 0$. The constants m, c, and e are the rates of immigration from the mainland, colonization, and extinction, respectively.

(a) Suppose that there is no immigration from the mainland: $m = 0$. Draw a complete bifurcation diagram using c as the bifurcation parameter relative to a fixed value of e and describe the type(s) of bifurcation(s) in the diagram.

(b) Suppose now that there is immigration from the mainland: $m > 0$. Show that there exists a unique positive fixed point and determine its stability.

8. Consider this model for the concentration of substrate, S, and the density of heterotroph, H, in a chemostat:

$$\frac{dS}{dT} = D(S_i - S) - \frac{\mu_1}{Y_1} SH$$

$$\frac{dH}{dT} = \mu_1 SH - DH$$

with D, S_i, μ_1 and Y_1 being positive constants.

(a) Describe the processes that this system models.

(b) Non-dimensionalize this system.

(c) Analyze the long term behavior of the dimensionless system.

No calculators, laptops, books or notes allowed. Be sure to respond to each part of each question, show all your work, and justify your answers. Quality counts more than quantity.

Print Name _____

1. Suppose a Galton-Watson branching process, $\{N_t\}_{t=0}^{\infty}$, has an offspring probability generating function given by:

$$F(x) = \frac{0.1}{1 - 0.9x}.$$

(a) Find the two probabilities p_0 (no children) and p_1 (one child).

(b) Find the mean number of offspring.

(c) Determine the long-term probability of extinction.

2. Consider the following model for annual plants

$$\frac{dV}{dt} = uV$$
$$\frac{dR}{dt} = (1 - u)V,$$

where V is the mass of the vegetative growth, R is the mass of the reproductive growth and u is the fraction of photosynthate devoted to vegetative growth, with $0 \leq u(t) \leq 1$ for $0 \leq t \leq T$.

Suppose we want to maximize the reproductive growth over the time range $[0, T]$, so we want to find

$$\max_u \int_0^T \ln R(t)\, dt.$$

(a) State the necessary conditions for this optimal control problem, assuming $V(0)$ and $R(0)$ are given.

(b) Describe the possible state trajectories that could be part of this solution. A complete solution connecting these trajectories for various parameter cases is not required.

3. Consider the following population model:

$$u_{t+1} = a\frac{u_t^2}{b^2 + u_t^2}, \qquad a > 0.$$

Determine the conditions under which the population can go extinct if it becomes less than a critical size and determine this critical size.

4. The Leslie Matrix for an age-structured population is given by

$$L = \begin{bmatrix} 0 & 2 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix},$$

with $s_2 > 0$.

   (a) Interpret the terms of this matrix with respect to assumptions about birth rates and survival rates for the various age classes.

   (b) Derive the Euler-Lotka equation for this system, and calculate the eigenvalues for this system.

   (c) For what condition(s) on s_2 does this system support long-term growth? Characterize the age distribution of the population for large t.

5. Consider a competition model for two species, N_1 and N_2, given by the following system:

$$\frac{dN_1}{dT} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)$$
$$\frac{dN_2}{dT} = r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2}\right),$$

where all the parameters, r_1, r_2, K_1, K_2, b_{12}, and b_{21}, are positive.

   (a) Verify that the above system can be non-dimensionalized into the following system:

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = ay(1 - bx)$$

   by specifying the new dimensionless parameters, a and b, in terms of the parameters of the original system.

   (b) Analyze the linear stability of all the biologically feasible equilibrium points of the new dimensionless system from part (a) and identify the type(s) of bifurcation(s) that can occur.

   (c) Using the information found in part (b), plot a bifurcation diagram of the x component of the fixed points as a function of a suitable bifurcation parameter.

2

6. For the (scaled) Fisher Equation:

$$u_t = u(1 - u) + u_{xx}, \quad -\infty < x < \infty, t > 0,$$

Derive the conditions on a heteroclinic solution connecting $u = 1$ as $x \to -\infty$ to $u = 0$ as $x \to \infty$. Show this solution is a traveling wave and find it's minimum speed.

7. Consider this population model:

$$\frac{\partial N}{\partial t} = rN + D\frac{\partial^2 N}{\partial x^2}$$

for $0 < x < L$ and $t > 0$ and given initial distribution $N(x, 0) = N_0(x)$ for $0 < x < L$. Assume r and D are positive constants.

(a) Describe the meaning of each type of boundary condition below in terms of the effect on the population:

$$(A) \quad N(0) \quad = N(L) = 0$$
$$(B) \quad N'(0) \quad = N'(L) = 0.$$

(b) Give an explicit solutions to this PDE using boundary condition (A) with initial condition $N_0(x) = 5\sin\left(\frac{\pi x}{L}\right)$.

Give an explicit solution using boundary condition (B) with initial condition $N_0(x) = 2 + \cos\left(\frac{5\pi x}{L}\right)$.

8. Consider the following assumptions:

   i. Species N_1 and N_2 both exhibit density dependent growth in the absence of any interaction and have respective carrying capacities, K_1 and K_2.

   ii. Species N_2 positively affects the growth of species N_1 in such a way that N_1 can potentially exceed its carrying capacity, K_1.

   iii. Species N_2 is harvested at a maximal rate of, h, and in a density dependent manner reflected by a Type II functional response.

   iv. Assume species N_1 can survive without species N_2.

(a) Construct a 2 species model that captures the above characteristics and state all assumptions on your model's parameters.

(b) Now, assume that h, the harvesting rate is equal to zero. By using the zero-growth isoclines (i.e nullclines) for N_1 and N_2 and vector field arguments, determine if the coexistence equilibrium is stable.

3

Math 581-582 Prelim Exam, January 2012

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

Print Name _____

1. The probability generating function of a branching (Galton-Watson) process with $N_0 = 1$ is
$$F(x) = ax^2 + bx + c,$$
where a, b and c are positive constants. Investigate the long-term probability of extinction, giving any needed assumptions on F and its coefficients.

2. Consider the following delay differential equation model for a laboratory population of blowflies:
$$\frac{dN}{dt} = \alpha N(t - \tau)e^{-\beta N(t-\tau)} - \delta N(t)$$
with α, β, τ and δ positive constants. Assume that $\alpha > \delta$.

What are the long-term equilibria for this model and discuss the stability of the extinction equilibrium $N^* = 0$.

3. Determine the asymptotic sex ratio for the model where $F(t)$ represents the number of females at time t and $M(t)$ represents the number of males at time t. Birth and death rates are given by positive constants, b_f, b_m and μ_f, μ_m respectively.

$$\frac{dF}{dt} = -\mu_f F + b_f \sqrt{FM}$$

$$\frac{dM}{dt} = -\mu_m M + b_m \sqrt{FM}$$

(Hint: Try change of variables $R^2(t) = F(t)$, $S^2(t) = M(t)$.)

4. a. Interpret the terms in this population model:

$$\frac{\partial N}{\partial t} = rN + b\frac{\partial N}{\partial x} + D\frac{\partial^2 N}{\partial x^2}$$

given initial distribution $N(x, 0) = N_0(x)$ for x in [0,L] and $t > 0$ and boundary conditions $N(0, t) = 0$ and $N(L, t) = 0$. Assume r, b and D are positive constants.

b. Derive the critical patch size for this model.

1

5. Consider the discrete population model,

$$x_{t+1} = \frac{ax_t}{b + x_t}$$

with a and b being positive constants. Does this model have any period-2 cycles? Why or why not?

6. This model represents a population $x(t)$, where $u(t)$ represents a harvest control.

$$x' = -ax - \frac{bu}{c + u}$$

with $x(0) = x_0$ and $x(T) = x_1$ with $x_0 > x_1$.

a. Write the necessary conditions to minimize

$$\int_0^T u(t)dt.$$

with $0 \leq u(t) \leq u_M$. (For convenience, assume $a = b = c = 1$.) State any conditions needed so that an admissible control exists.

b. With $a = b = c = 1$, solve for the optimal control and corresponding state.

7. Consider this epidemic model with $x(t)$ = the number of susceptibles individuals, $y(t)$ = the number of infected individuals, and $z(t)$ = the number of individuals who have died from the disease by time t. This epidemic occurs rapidly so we can ignore changes in population due to birth, natural death, or emigration.

$$x' = -kxy$$

$$y' = kxy - \gamma y$$

$$z' = \gamma y.$$

Reduce this system to one equation in z only. Then non-dimensionalize this z equation and then analyze the stability of the fixed points of the non-dimensionalized differential equation.

8. Consider this Leslie matrix for a population with two stages.

$$L = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$$

a. Explain why this matrix is imprimitive.

b. Explain the long-term behavior of this population.

c. Given initial conditions $x_1 = 10$ and $x_2 = 10$, find explicitly the form of the solution of this system.

2

Math Ecology Prelim Exam, August 2011

No calculators, laptops, books or notes allowed. Show all of your work. Justify your answers.

Print Name _____

1. Suppose that each parent has exactly three children, and that each newborn is equally likely to be male or female. Start with a single male parent and assume that the number of male progeny can be modeled as a branching process. Calculate the long time probability of extinction for this male lineage.

2. Suppose the growth (or decay) of population with biocontrol rate U is given by

$$X' = \alpha X \ln(\frac{\theta}{X}) - \frac{k_1 U X}{k_2 + U}$$

with $X(0) = X_o > X(T) = X_1 > 0$, and T is the fixed final time. The positive parameter θ is the plateau size which an uncontrolled population would approach as $t \to \infty$. The parameters, k_1, k_2 and α, are positive. Assume $\theta > X_o$. The control U satisfies $0 \le U(t) \le U_{max}$.

a. Change variables to make the state differential equation dimensionless. Note that a nonlinear scaling of the state variable may simplify the problem.

b. Write the necessary conditions using the transformed state equation with appropriate rescaled control u, and final time t_1 to minimize

$$\int_0^{t_1} u(t)dt$$

with $0 \le u(t) \le u_{max}$. State any conditions needed so that an admissible control exists.

c. State conditions on the parameters to give the structure of the optimal control and corresponding state.

3. Consider this population model:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(N)n \tag{1}$$

$$n(0,t) = \int_0^\infty m_0 a e^{-\gamma a} n(a,t)\,da \tag{2}$$

$$n(a,0) = n_0(a) \tag{3}$$

$$N(t) = \int_0^\infty n(a,t)\,da. \tag{4}$$

a. Functions μ and n_o are non-negative and constants γ and m_o are positive. Interpret the terms of this model.

b. Let

$$G(t) = \int_0^\infty m_0 e^{-\gamma a} n(a,t)\,da$$

and

$$B(t) = \int_0^\infty m_0 a e^{-\gamma a} n(a,t)\,da.$$

Derive a system of three differential equations for $N(t)$, $G(t)$ and $B(t)$.

4. A Leslie matrix for an age-structure model is given by

$$L = \begin{bmatrix} 0 & 0 & 6a^3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$$

with $a > 0$.

a. Interpret the terms of the matrix.

b. Find the characteristic equation and the eigenvalues.

c. Show that the matrix is imprimitive.

d. Give any conclusions about this population that you can find from these calculations.

5. Discuss the qualitative behavior of this population system by examining its equilibrium points and corresponding stability. Assume that $x(t) \geq 0$ and $y(t) \geq 0$.

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y(.75 - y - .5x)$$

6. a. Interpret the various terms in this population model. Give appropriate assumptions about the functions involved.

$$B(t) = \int_0^t B(t-a)l(a)m(a)da + \int_0^{w-t} n(a,0)\frac{l(a+t)}{l(a)}m(a+t)da.$$

   b. Derive the Euler-Lotka equation (characteristic) for this population model.

   c. Prove why the Euler-Lotka equation has exactly one real root.

   d. What is the relationship between this root and the net reproductive rate?

7. Consider this model for population on an infinite spatial domain,

$$\frac{\partial u}{\partial t} = ru + D\frac{\partial^2 u}{\partial x^2}$$

   for all x on the real line and $t > 0$ with positive constants r and D.
   What is the relationship between long term average velocity of expansion (starting with a point source) and the constants r and D? How is the relationship derived?

8. Consider this discrete time model for a population density.

$$x_{t+1} = g(x_t)x_t$$

   a. Suppose x^* is an equilibrium point of this equation. How is local stability of this equilibrium point determined from properties of the function g? Give needed assumptions on g.

   b. For $g(x) = e^{-x} + \frac{1}{2}$, determine the equilibrium point(s) and analyze stability.

SOME USEFUL RESULTS

Routh-Hurwitz Stability Critiera

Given a polynomial equation of degree m of the form,

$$\lambda^m + a_1 \lambda^{m-1} + a_2 \lambda^{m-2} + \cdots + a_m = 0,$$

the Routh-Hurwitz stability criteria for $m=2$ and $m=3$ are

$m = 2$: $\qquad\qquad\qquad a_1 > 0; \; a_2 > 0$.

$m = 3$: $\qquad\qquad\qquad a_1 > 0; \; a_3 > 0; \; a_1 a_2 > a_3$.

Jury Test

Given a polynomial equation of degree 2 of the form

$$\lambda^2 + a_1 \lambda + a_2 = 0,$$

the Jury conditions can be written

$$1 + a_1 + a_2 > 0,$$

$$1 - a_1 + a_2 > 0,$$

$$|a_2| < 1.$$

Math Ecology Prelim Exam, January 2011

No calculators, laptops, books or notes allowed. Justify your answers.

Print Name _____

1. Analyze the stability of the equilibrium points of this model with $r, T, K > 0$.

$$\frac{dx(t)}{dt} = rx(t - T)\left(1 - \frac{x(t)}{K}\right)$$

2. Consider a birth and death process with constant immigration, with probabilities

$$P[\Delta N = +1 \,|\, N(t) = n] = \beta n \Delta t + I \Delta t + o((\Delta t)^2).$$

$$P[\Delta N = -1 \,|\, N(t) = n] = \mu n \Delta t + o((\Delta t)^2).$$

$$P[\Delta N = 0 \,|\, N(t) = n] = 1 - [(\beta + \mu)n + I]\Delta t + o((\Delta t)^2).$$

Assume that the mortality rate is greater than the birth rate, $\mu > \beta$.

a. Derive a system of differential equations for the probabilities $p_n(t) = P[N(t) = n]$, Give appropriate initial conditions.

b. Derive a partial differential equation for the probability generating function for the process $N(t)$.

3. Consider the following population model.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{bx}{a + x},$$

where r, K, a and b are positive parameters.

a. Briefly explain the assumptions underlying this model for a population with density x.

b. Non-dimensionalize this model and then determine how the stability and number of equilibria change as the parameters change.

4. Consider Fisher's equation

$$\frac{\partial u}{\partial t} = u(1 - u) + \frac{\partial^2 u}{\partial x^2}$$

on an infinite spatial domain and $t > 0$.

a. Assume a rightward moving traveling wave solution and transform this PDE into an ODE.

b. Do a phase plane analysis of this ODE. Characterize each of the equilibria in the phase plane.

c. Determine the minimum speed of the traveling wave.

5. Contrast the nature of stationary solutions for Fisher's equation

$$\frac{\partial u}{\partial t} = ru + D\frac{\partial^2 u}{\partial x^2}$$

on a finite spatial domain $[0, L]$ and $t > 0$, with Dirichet boundary conditions $(u(0, t) = u(L, t) = 0)$ and with Neumann boundary conditions $(u_x(0, t) = u_x(L, t) = 0)$. Assume $u(x, 0) = u_o(x)$ and r and D are positive constants. Explain the differences in the biological meaning of these 2 types of boundary conditions.

6. Consider the following S, I, R epidemic model

$$S_{t+1} = S_t - \frac{\beta}{N}I_t S_t + b(I_t + R_t)$$

$$I_{t+1} = I_t(1 - \gamma - b) + \frac{\beta}{N}I_t S$$

$$R_{t+1} = R_t(1 - b) + \gamma I_t.$$

where $N = S_0 + I_0 + R_0$ with S_0, I_0, R_0 being positive and $0 < \gamma$ and $0 < b$ and

$$0 < \beta < 1$$

$$0 < b + \gamma < 1.$$

a. Give the interpretation of this model by explaining each term.

b. Find all the equilibrium points for this model and discuss the stability of the disease free equilibrium.

7. Consider this model of above ground biomass of forest trees F, savanna trees S, and understory grass layer G,

$$\frac{dF}{dt} = r_f F(1 - \frac{F}{K_f}) - M_f FL$$

$$\frac{dS}{dt} = r_s S(1 - \frac{S}{K_s} - \frac{F}{K_f}) - M_s SL$$

$$\frac{dG}{dt} = r_g G(1 - \frac{S}{K_s} - \frac{G}{K_g} - \frac{F}{K_f}) - M_g GL$$

The parameters, $K_f, K_s, K_g, M_f, M_s, M_g, L, r_f, r_s, r_g$, are positive.

a. Change variables to nondimensionalize this system.

b. Derive conditions under which the equilibrium with only forest trees present is locally stable.

8. Derive the Euler-Lotka equation for this McKendrick-von Foerster PDE,

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a)n$$

with

$$n(0, t) = \int_0^\omega n(a, t)m(a)\,da$$

and

$$n(a, 0) = n_o(a),$$

with m, μ and n_o being non-negative functions.

Math 581-582 Prelim Exam, Fall 2010

No calculators, laptops, books or notes allowed. Work as many of the following problems as you are able. Be sure to respond to each part of each question and show all of your work. Quality counts more than quantity. Justify your answers.

Print Name _____

1. Consider this model for the spread of a disease, where the populations are susceptibles S and infecteds I.

$$S'(t) = \theta - dS - \beta SI + \gamma I$$
$$I'(t) = \beta SI - (d + \nu + \gamma)I$$

a. Give the interpretation of this model by discussing the assumptions inherent in each term in the right hand side. Assume

$$\theta, \beta, \gamma, \nu, d$$

are positive constants.

b. Find biologically relevant equilibrium points for this model. Analyze the stability of those points. Be certain to specify any assumptions necessary on the parameters arising from your stability analysis.

2. Consider a population growing exponentially in which time has been rescaled so that $x'(t) = x(t)$. Let the control u be a source term that increases the population level. We seek to maximize the population over time, while minimizing the cost of implementing this control.

$$\max_{u} \int_0^2 [2x(t) - (3u(t) + u^2(t))]dt$$

subject to

$$x'(t) = x(t) + u(t)$$

and $x(0) = 5$ and $0 \leq u(t) \leq 2$

Find the optimal control and solve the corresponding state and adjoint equations.

3. A Leslie matrix for an age-structure model is given by

$$L = \begin{bmatrix} 0 & \frac{3a^2}{2} & \frac{3a^3}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$$

with $a > 0$.

a. Interpret the terms of the matrix.

b. Find the eigenvalues and determine the dominant eigenvalue and its eigenvector.

c. Determine the stable age distribution.

d. Show that L^k is a positive matrix for some k.

4. a. Derive the critical patch size for this model of a population $N(x, t)$.

$$\frac{\partial N}{\partial t} = rN + D\frac{\partial^2 N}{\partial x^2}$$

given initial distribution $N(x, 0) = N_0(x)$ for x in [0,L] and $t > 0$ and boundary conditions $N(0, t) = 0$ and $N(L, t) = 0$.

b. How is the critical patch size affected if the PDE was

$$\frac{\partial N}{\partial t} = rN + \frac{\partial N}{\partial x} + D\frac{\partial^2 N}{\partial x^2}$$

with the same boundary and initial conditions?

5. Consider the discrete population model,

$$x_{t+1} = \frac{ax_t}{b + x_t}$$

with a and b being positive constants.

a. Find its biologically feasible equilibria and analyze the stability of those equilibria.

b. Are there any period 2-cycles? Why?

6. Consider this reaction-diffusion model:

$$\frac{\partial n}{\partial t} = f(n) + \frac{\partial^2 n}{\partial x^2}$$

with

$$f(n) = n \text{ for } 0 \le n \le \frac{1}{2}$$

and

$$f(n) = 1 - n \text{ for } \frac{1}{2} \le n \le 1.$$

Shift to traveling wave coordinates. Solve the corresponding ordinary differential equation to determine the exact shape of the wave that satisfies $n(-\infty) = 1$ and $n(+\infty) = 0$.

7. A population of 100 ducks lives on two ponds, one large and one small. Let $N(t)$ be the number of ducks on the small pond. You may assume the number of ducks on the large pond is $100 - N(t)$. Let the probability of a departure from the small pond in a short time interval be proportional to the departure rate r_d, to the interval size, and to the number of birds on the small pond,

$$Pr[N(t + \Delta t) = n - 1 \mid N(t) = n] = r_d n \Delta t + o(\Delta t).$$

Similarly, assume that the probability of an arrival onto the small pond in a short time interval is proportional to the arrival rate r_a, to the interval size, and to the number of birds off the small pond,

$$Pr[N(t + \Delta t) = n + 1 \mid N(t) = n] = r_a(100 - n)\Delta t + o(\Delta t).$$

a. Derive a system of differential equations for $p_n(t) = P[N(t) = n]$, the probability of n birds on the small pond at time t.

b. Derive a partial differential equation for the probability generating function.

8. To consider how a small animal's body temperature depends on its environment, let θ be the animal's temperature in degrees (Celsius). (Think of a small ecotherm like a lizard.) Let the temperature in its microhabitat be T (degrees Celsius) where we asssume T is constant. The animal receives heat transfer with its microhabitat and radiative heating from solar radiation (with rate q in calories/hour). The differential equation for θ is

$$\frac{d\theta}{dt} = \frac{q}{mc} - \frac{k}{mc}(\theta - T).$$
$$\theta(0) = \theta_0$$

Also m is mass in grams, c is the specific heat of the animal in calories/((gram)(degree)) and k is the transfer coefficient in calories/((degree)(hour)).

a. Change variables to convert to a dimensionless form of the equation. Be sure to specify any new dimensionless parameters.

b. What is the limit of $\theta(t)$ as $t \to \infty$?

c. Temperatures T are not constant in reality, so suggest how you would modify this model to account for diurnal (within day) temperature variation. For your suggested modification, discuss the response of body temperature to environmental temperature fluctuations as dependent upon the time-scales of the environmental change to as compared to the time-scales of organism response, noting any dependence upon model parameters.