UT, Knoxville **Stochastics Preliminary Exam** **August 16, 2024**

Name: ______________________________

Problem:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	12	16	14	16	12	14	16	120
Score:										

1. (10 points) Let $\{A_n\}_{n\geq 1}$ be a sequence of independent events such that for all n, $\mathbb{P}(A_n) < 1$. Prove that the followings are equivalent

   (a) $\mathbb{P}(\bigcup_{n\geq 1} A_n) = 1$;

   (b) $\mathbb{P}(A_n \text{ i.o.}) = 1$

2. (10 points) Let $X \leq Y \leq Z$ be real random variables such that X and Z are integrable. Show that Y is integrable.

3. (12 points) Let $\{X_n\}_{n\geq 1}$ be i.i.d. symmetric Bernoulli random variables. Determine the distribution of Y given by

$$Y = \sum_{n=1}^{\infty} \frac{X_n}{2^n}.$$

   [Hint: You may use the identity $\cos\alpha = \frac{\sin 2\alpha}{2\sin\alpha}$.]

4. (16 points) Let X_n be i.i.d. Weibull distributed random variables with density $f(x) = 3x^2\exp(-x^3)$ for $x \geq 0$ and $f(x)$. Prove that

$$\limsup_{n\to\infty} \frac{X_n}{(\log n)^{1/3}} = 1 \quad a.s.$$

   Deduce that

$$\frac{\max\{X_1, \ldots, X_n\}}{(\log n)^{1/3}} \to 1 \quad a.s. \quad \text{as } n \to \infty.$$

5. (14 points) Let $\{X_n\}_{n\geq 1}$ be random variables with distribution functions $\{F_n\}_{n\geq 1}$, respectively. Suppose that $X_n \xrightarrow{d} X$ as $n \to \infty$, where X is a random variable having continuous distribution function $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. Prove that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0 \quad \text{as } n \to \infty.$$

6. (16 points) Let $X_n, n \geq 1$, be independent random variables such that for each n, X_n has the uniform distribution on $(0, n)$, $n \geq 1$. Determine a_n such that

$$\frac{X_1^2 + \cdots + X_n^2}{a_n} \to 1 \quad \text{in probability as } n \to \infty.$$

[Hint: Notice that X_n are not equally distributed but $Y_n := n^{-1}X_n$ are i.i.d.(why?), and conversely, $X_n = nY_n$. You may also need $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.]

7. (12 points) Let $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ be such that for some sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$

$$\mathbb{E}(X \,|\, \mathcal{G}) = Y \quad \text{and} \quad \mathbb{E}(X^2 \,|\, \mathcal{G}) = Y^2 \quad \text{a.s.}$$

Prove that $X = Y$ a.s.

8. (14 points) Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration.
For each $n \geq 0$ choose $A \in \mathcal{F}_n$ and define

$$\tau_n = (n+1)\mathbf{1}_A + n\mathbf{1}_{A^c}$$

Prove that τ_n is a bounded stopping time.

9. (16 points) Let $\{X_n\}_{n\geq 0}$ be a stochastic process adapted to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ such that for any bounded stopping time τ

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0).$$

Prove that $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a martingale.

[Hint: Substitute τ_n for τ in the above equality of expectations.]

Probability Prelim, 2023

1. (10 points). Let $\{X_n\}$ be a sequence of independent random variables such that for each $n \geq 1$, X_n is exponentially distributed with the density

$$f_n(x) = \lambda_n e^{-\lambda_n x} \qquad x > 0$$

where $\lambda_n > 0$.

Prove that with probability 1,

$$\liminf_{n\to\infty} X_n$$

is either 0 or ∞, and find the condition (in terms of $\{\lambda_n\}$) for each case.

2. (10 points). Let $\{X, X_k\}_{k\geq 1}$ be an i.i.d. sequence. Prove that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} < \infty \qquad a.s.$$

if and only if

$$E\exp\{\theta X\} < \infty$$

for some $\theta > 0$.

3. (10 points). Let $\{X, X_k\}_{k\geq 1}$ be an i.i.d. sequence with $EX^2 < \infty$. Prove that

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^n kX_k = \frac{1}{2}EX \qquad a.s.$$

4. (10 points). Given two random variables X and Y such that $E|X| < \infty$, $E|Y| < \infty$ and

$$EX1_A = EY1_A \qquad \forall A \in \sigma(X, Y)$$

Prove that $X = Y$ a.s.

5. (10 points). Let $0 < p < 1$. Construct a probability model to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{[np]} \binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{2}$$

where $[np]$ is the integer part of np.

6. (10 points). Given a random variable X with $EX^2 < \infty$ and two sub σ-algebras $\mathcal{G}_1 \subset \mathcal{G}_2$, prove that

$$E\Big(Var(X|\mathcal{G}_1)\Big) \geq E\Big(Var(X|\mathcal{G}_2)\Big)$$

7. (10 points). Let $\{X_k\}$ be a sequence of random variables (not necessarily independent) such that $EX_k^2 < \infty$ $(k = 1, 2, \cdots)$. Assume that

$$\lim_{n\to\infty} \sum_{k=1}^{n} EX_k = \infty$$

and

$$\lim_{n\to\infty} \sum_{j,k=1}^{n} Cov(X_j, X_k) \Big/ \Big(\sum_{k=1}^{n} EX_k\Big)^2 = 0$$

Prove that

$$\sum_{k=1}^{n} X_k \Big/ \sum_{k=1}^{n} EX_k \xrightarrow{P} 1 \qquad (n \to \infty)$$

8. (20 points). Let $\{X, X_k\}_{k\geq 1}$ be an i.i.d. sequence with the common distribution $P\{X = -1\} = P\{X = 1\} = 1/2$. Set

$$S_0 = 0 \ \text{ and } \ S_n = X_1 + \cdots + X_n \qquad n = 1, 2, \cdots$$

(1) Prove that for any $\theta > 0$,

$$M_n = (\cosh\theta)^{-n} \exp\{\theta S_n\} \qquad n = 0, 1, \cdots$$

is a martingale under the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ given as

$$\mathcal{F}_0 = \{\phi, \Omega\} \ \text{ and } \ \mathcal{F}_n = \sigma\{S_1, \cdots, S_n\} \qquad n = 1, 2, \cdots$$

Here we recall that $\cosh x = \frac{e^x + e^{-x}}{2}$.

(2). Define $\tau = \min\{n \geq 1; \ \ S_n = 2023\}$ in the covention that $\tau = \infty$ if $S_n \neq 2023$ for all $n \geq 1$. Prove that for any $\theta > 0$,

$$E(\cosh\theta)^{-\tau} 1_{\{\tau<\infty\}} = \exp\{-2023\theta\}$$

and derive that $\tau < \infty$ a.s.

9. (10 points). Let $\{X, X_k\}_{k\geq 1}$ be an i.i.d. sequence with $EX = 0$ and $EX^2 < \infty$ and set

$$S_0 = 0 \ \text{ and } \ S_n = X_1 + \cdots + X_n \qquad n = 1, 2, \cdots$$

Let τ be a stopping time with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ given as

$$\mathcal{F}_0 = \{\phi, \Omega\} \ \text{ and } \ \mathcal{F}_n = \sigma\{S_1, \cdots, S_n\} \qquad n = 1, 2, \cdots$$

such that $E\tau < \infty$. Prove that

$$E \max_{1\leq k\leq\tau} S_k^2 \leq 4E\tau \cdot EX^2$$

Name: ____

1. (10 points; 5 points each) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1 \in \mathcal{F}, A_2 \in \mathcal{F}, \ldots$. Show the following inequalities:

   (i) For any $n \geqslant 2$,
   $$P(\bigcup_{k=1}^{n} A_k) \geqslant \sum_{k=1}^{n} P(A_k) - \sum_{1 \leqslant i < j \leqslant n} P(A_i \cap A_j)$$

   (ii) For any $n \geqslant 3$,
   $$P(\bigcup_{k=1}^{n} A_k) \leqslant \sum_{k=1}^{n} P(A_k) - \sum_{1 \leqslant i < j \leqslant n} P(A_i \cap A_j) + \sum_{1 \leqslant i < j < k \leqslant n} P(A_i \cap A_j \cap A_k)$$

2. (10 points) Let $X_1, X_2, \ldots$ be i.i.d random variables such that $P(X_1 = 1) = \frac{1}{2}$ and $P(X_1 = 2) = \frac{1}{2}$. Show that
   $$\lim_{n \to \infty} \frac{X_1^3 + X_2^3 + \ldots + X_n^3}{X_1^2 + X_2^2 + \ldots + X_n^2} = \frac{9}{5} \quad \text{a.s.}$$

3. (10 points) Prove that
   $$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2} \, .$$

4. (10 points) Suppose that $\{X_k, k \geqslant 1\}$ are random variables such that
   $$P(X_k = -k^2) = \frac{1}{k^2}, \quad P(X_k = -k^3) = \frac{1}{k^3}, \quad P(X_k = 2) = 1 - \frac{1}{k^2} - \frac{1}{k^3} \, .$$
   Prove that $\sum_{k=1}^{n} X_k \stackrel{a.s.}{\to} +\infty$ as $n \to \infty$.

5. (10 points) Let $\{X_n\}_{n \geqslant 1}$ be a sequence of i.i.d. random variables with
   $$P(X_n = 1) = P(X_n = -1) = 1/2 \, .$$
   What can be concluded about the probability
   $$P\left(\sum_{n=1}^{\infty} \frac{X_n}{n} \text{ converges} \right) ?$$

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6. (10 points) Let $X, Y \in L^1$ be two random variables, and $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ an increasing sequence of σ-algebras. Suppose
$$E[I_A X] = E[I_A Y]$$
holds for any $A \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Prove that then
$$E[I_A X] = E[I_A Y]$$
also holds for all $A \in \mathcal{F}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n)$.

7. (10 points) Let X, Y, and Z be three integrable (but not square-integrable) random variables. Suppose we have
$$E[X|Y] = Z,\ E[Y|Z] = X,\ \text{and}\ E[Z|X] = Y.$$
Show that $X = Y = Z$ a.s.

8. (10 points) In a Bernoulli trial of gambling games, the probability that the gambler wins in a given game is $0 < p < 1$. Each time he loses, he loses one dollar. The rewarded money for his winning games form an i.i.d. sequence of positive random variables with common expectation equal to 0.8 dollar. For each $n \in \mathbb{N}$, let W_n be the gambler's total winning (we count the loss as negative winning) after his n-th win. Find the almost sure limit
$$\lim_{n\to\infty} \frac{W_n}{n} .$$

9. (10 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $Y \in L^1$, and $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a filtration. Set
$$X_n := E[Y|\mathcal{F}_n], \quad n \in \mathbb{N} .$$
Let τ be a stopping time such that $\tau < \infty$ a.s.
   (i) Show that $\{(X_n, \mathcal{F}_n)\}_{n\in\mathbb{N}}$ is a martingale.
   (ii) Prove Doob's optional sampling theorem, that is, $\{(X_\tau, \mathcal{F}_\tau), (Y, \mathcal{F})\}$ is a martingale.
   (iii) Show that for any stopping time $\tilde{\tau}$ with $\tilde{\tau} < \infty$ a.s.,
$$E[X_\tau] = E[X_{\tilde{\tau}}] .$$

Name: ____

1. (10 points) Prove the following generalization of subadditivity: For any events $A_i \subset B_i$, $i \in \mathbb{N}$, in a probability space $(\Omega, \mathcal{F}, P)$,
$$P\Big(\bigcup_{i\in\mathbb{N}} B_i\Big) - P\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \leqslant \sum_{i\in\mathbb{N}} \Big(P(B_i) - P(A_i)\Big) .$$

2. (10 points) Show that if $X_1, X_2, \ldots$ are i.i.d., non-degenerate (i.e. X_1 is not equal to a constant a.s.) random variables, then
$$P(X_n \text{ converges}) = 0 .$$
Hint: Use Kolmogorov's zero-one law and Borel-Cantelli lemma.

3. (10 points) Prove the following statement: If there exists an $\epsilon > 0$ such that $P(A_n) \geqslant \epsilon$ for infinitely many $n \in \mathbb{N}$, then we have $P(A_n \text{ i.o.}) \geqslant \epsilon$.

4. (10 points) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables such that $E[X_n] = 0$ for all $n \in \mathbb{N}$,
$$\sum_{n=1}^{\infty} E\Big[|X_n| \mathbb{1}\{|X_n| > 1\}\Big] < \infty \text{ and } \sum_{n=1}^{\infty} E\Big[X_n^2 \mathbb{1}\{|X_n| \leqslant 1\}\Big] < \infty .$$
Show that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

5. (10 points) Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with the common distribution function F, such that
$$\sup\{x \in \mathbb{R} : F(x) < 1\} = +\infty ,$$
and let
$$\tau(t) = \min\{n : X_n > t\}, \quad t > 0,$$
that is, $\tau(t)$ is the index of the first X-variable that exceeds the level t. Show that
$$p_t \tau(t) \xrightarrow{d} \mathrm{Exp}(1) \quad \text{as } t \to \infty$$
where $p_t = P(X > t)$.

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6. (10 points)

   (i) Let X and Y be two independent standard normal random variables. Use the method of characteristic function to show that

   $$X_1 = \frac{X+Y}{\sqrt{2}} \quad \text{and} \quad X_2 = \frac{X-Y}{\sqrt{2}}$$

   are independent standard normal random variables.

   (ii) Let X be a standard normal random variable and let the random variable ξ be independent of X and have the distribution $P(\xi = -1) = P(\xi = 1) = 1/2$. Set $Y = \xi X$. Prove that X and Y are uncorrelated but dependent standard normal random variables.

7. (10 points) Suppose that Y is a random variable with finite variance and that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Recall that $\text{Var}(Y|\mathcal{G}) = E[(Y - E[Y|\mathcal{G}])^2|\mathcal{G}]$. Prove that

$$\text{Var}(Y) = E[\text{Var}(Y|\mathcal{G})] + \text{Var}(E[Y|\mathcal{G}]) \ .$$

8. (10 points) Let X, Y, and Z be three square-integrable random variables. Suppose we have

$$E[X|Y] = Z, \ E[Y|Z] = X, \text{ and } E[Z|X] = Y.$$

Show that $X = Y = Z$ a.s.

9. (10 points) Let $Y, Y_1, Y_2, \ldots$ be i.i.d. random variables with the common distribution $P(Y = -1) = P(Y = 1) = \frac{1}{2}$. Set

$$S_0 = 0 \quad \text{and} \quad S_n = Y_1 + \ldots + Y_n, \quad n = 1, 2, \ldots \ .$$

Prove that the sequence

$$X_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n \ , \quad n = 0, 1, 2, \ldots$$

is a martingale with respect to the natural filtration

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \mathcal{F}_n = \sigma\{Y_1, \ldots, Y_n\} \ , \quad n = 1, 2, \ldots \ .$$

10. (10 points) Let $X, X_1, X_2, \ldots$ be random variables with $X_n \to X$ in L^1. Show that for any increasing σ-algebras $\mathcal{F}_n$,

$$E[X_n|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ in } L^1 \text{ as } n \to \infty \ .$$

Probability Prelim, January 18, 2022

1. (12points). Let $\{A_n\}$ be a sequence of events. Show that

$$P\Big(\liminf_{n\to\infty} A_n\Big) \le \liminf_{n\to\infty} P(A_n)$$

and

$$P\Big(\limsup_{n\to\infty} A_n\Big) \ge \limsup_{n\to\infty} P(A_n)$$

2. (14points). Let $\{X_n\}_{n\ge 1}$ be a sequence of random variables with the distributions

$$P\{X_n = 0\} = 1 - \frac{1}{2^n} \quad\text{and}\quad P\{X_n = \pm 2^n\} = \frac{1}{2^{n+1}} \qquad n = 1, 2, \cdots$$

(a). Prove that the sequence almost surely converges and find the limit.

(b). Does X_n converge in L_1? (To receive credit, you have to prove your conclusion).

3. (12 points). Let $\{X, X_n\}_{n\ge 1}$ be an i.i.d. sequence of random variables and assume that there is a $\lambda > 0$ such that

$$E\exp\Big\{\theta X\Big\} = \begin{cases} \text{finite} & \forall\theta < \lambda \\ \infty & \forall\theta > \lambda \end{cases}$$

Prove that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = \lambda^{-1} \qquad a.s.$$

4. (12 points). Let $\{X_n\}_{n\ge 1}$ be a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$. Prove that the limit

$$\lim_{n\to\infty} (X_1 X_2 \cdots X_n)^{1/n}$$

exists almost surely and compute its value. (Hint: find $E\ln(X_1)$)).

5. (10 points). Use the central limit theorem to prove that

$$\lim_{n\to\infty} e^{-n} \sum_{k=n+1}^{\infty} \frac{n^k}{k!} = \frac{1}{2}$$

6. (14 points). (a). Let $\{X_n\}_{n\ge 1}$ be a sequence of non-negative random variables such that $X_n \ge X_{n+1}$ for any $n \ge 1$. Assume that $X_n \xrightarrow{P} 0$. Prove that $X_n \xrightarrow{a.s.} 0$

(b) Prove that for a monotonic (non-decreasing or non-increasing) random sequence X_n, $X_n \xrightarrow{P} X$ if and only if $X_n \xrightarrow{a.s.} X$

7. (12 points). Let X and Y be independent, identically distributed random variables with finite mean. Prove that

$$E[X|X+Y] = \frac{X+Y}{2}$$

8. (14 points). Let $\{X_n\}_{n\geq 1}$ be an i.i.d. sequence of standard normal random variables and set

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n \qquad n = 1, 2, \cdots.$$

(a). Prove that

$$M_n = \exp\left\{S_n - \frac{n}{2}\right\} \qquad n = 0, 1, 2, \cdots$$

is a martingale in connection to the filtration

$$\mathcal{F}_0 = \{\Omega, \phi\} \quad \text{and} \quad \mathcal{F}_n = \sigma\{X_1, \cdots, X_n\} \qquad n = 1, 2, \cdots.$$

(b). Prove that for any integer $n \geq 1$,

$$E \exp\left\{\max_{1\leq k\leq n} (2S_k - k)\right\} \leq 4e^n$$

Probability Prelim August 13, 2021

1. (12 pts) Let the integer $n \geq 2$ and $A_1, \cdots, A_n$ be events in a probability space in $(\Omega, \mathcal{F}, P)$.

(a). Show that

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i=2}^{n} P(A_1 \cap A_i).$$

(b). Deduce the following improvement on the sub-additivity

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i) - \max_{1\leq k\leq n} \sum_{i\neq k}^{n} P(A_k \cap A_i).$$

2. (10 pts) Let $\{X, X_n\}_{n\geq 1}$ be an i.i.d. sequence of non-negative random variables with the common distribution function $F_X(x)$ satisfying

$$\lim_{x\to 0^+} \frac{F_X(x)}{x} = \lambda$$

for some $\lambda > 0$. Prove that the sequence

$$n \min_{1\leq k\leq n} X_k \qquad n = 1, 2, \cdots$$

converges in distribution and identify the limiting distribution.

3. (11 pts) Let $\{X_n\}_{n\geq 1}$ be an independent sequence of integer-valued random variables. Prove that the series

$$\sum_{n=1}^{\infty} X_n$$

converges almost surely if and only if

$$\sum_{n=1}^{\infty} P\{X_n \neq 0\} < \infty.$$

4. (10 pts) Let $\{X_n\}_{n\geq 1}$ be an i.i.d. sequence of standard normal random variables and set

$$Z_n = \exp\Big\{\sum_{k=1}^{n} X_k - \frac{n}{2}\Big\} \qquad n = 1, 2, \cdots.$$

(a). Prove that Z_n converges almost surely and identify the limit.

(b) Does Z_n converge in L_1? (To receive the credit, you have to prove your conclusion).

5. (12 pts) (a). Let X and Y be two independent standard normal random variables. Use the method of characteristic function to show that

$$X_1 = \frac{X+Y}{\sqrt{2}} \quad \text{and} \quad X_2 = \frac{X-Y}{\sqrt{2}}$$

are independent standard normal random variables.

(b) Let X be a standard normal random variable and let the random variable ξ be independent of X and have the distribution $P\{\xi = -1\} = P\{\xi = 1\} = 1/2$. Set $Y = \xi X$. Prove that X and Y are uncorrelated but <u>dependent</u> standard normal random variables.

6. (10 pts)Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. positive random variables with $EX_1 = \mu$ and $\sigma^2 = \text{Var}(X_1) < \infty$. Set

$$S_n = \sum_{k=1}^{n} X_k \qquad n = 1, 2, \cdots.$$

Prove that

$$\sqrt{S_n} - \sqrt{n\mu} \xrightarrow{d} N\big(0, \sigma^2/(4\mu)\big) \qquad (n \to \infty).$$

7. (10 pts) Let X and Y be random variables defined on a common probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-field. Show that if $E[Y|\mathcal{G}] = X$ and $E[X^2] = E[Y^2] < \infty$, then $X = Y$ a.s.

8. (10 pts) Let X_n be a non-negative martingale. Prove that for any stopping time τ with $\tau < \infty$ a.s., and any integer $k \geq 1$,

$$EX_{\tau+k} = EX_\tau.$$

9. (15 pts) Let $\{X, X_n\}_{n\geq 1}$ be an i.i.d. sequence with the common distribution $P\{X = -1\} = P\{X = 1\} = \frac{1}{2}$. Set

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n \qquad n = 1, 2, \cdots.$$

and let $a, b > 0$ be integers. Set

$$\tau_1 = \min\{n \geq 1; \quad S_n = -a\} \quad \text{and} \quad \tau_2 = \min\{n \geq 1; \quad S_n = b\}.$$

(i). Prove that $E(\tau_1 \wedge \tau_2) < \infty$.

(ii). Compute $P\{\tau_1 < \tau_2\}$ and $P\{\tau_1 > \tau_2\}$.

(iii). Compute $E(\tau_1 \wedge \tau_2)$.

(Hint for (i), (ii), (iii): Wald's equations).

UT, Knoxville **Stochastics Preliminary Exam** **August 12, 2020**

Name: ______________________________

1. (10 points) Prove the following generalization of subadditivity: For any events $B_i \subset A_i$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{P}\big(\bigcup_i A_i\big) - \mathbb{P}\big(\bigcup_i B_i\big) \le \sum_i \big(\mathbb{P}(A_i) - \mathbb{P}(B_i)\big)$$

provided $\sum_i \mathbb{P}(A_i) < \infty$.

2. (14 points) Recall that $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ is the L^p-norm of a random variable X, $p \in [1, \infty)$, and $\|X\|_\infty = \inf\{c \ge 0 : \mathbb{P}(|X| \le c) = 1\}$ is the L^∞-norm of X. Show that
(a) the function $[1, \infty) \ni p \mapsto \|X\|_p$ is nondecreasing;
(b) $\lim_{p\to\infty} \|X\|_p = \|X\|_\infty$.

3. (16 points) Let X_n be i.i.d. exponential random variables with mean 1. Prove that

$$\frac{\max\{X_1, \ldots, X_n\}}{\ln n} \to 1 \quad a.s. \quad \text{as } n \to \infty.$$

[Hint: First show that $\limsup_{n\to\infty} \frac{X_n}{\ln n} = 1$ a.s.]

4. (14 points) Let X_n, $n \ge 1$ be random variables and let $S_n = X_1 + \cdots + X_n$. Show that $\{n^{-1}S_n\}$ is uniformly integrable under each of the following two conditions
(a) X_n are uncorrelated and have the same mean and the same variance for every n;
(b) X_n are i.i.d. with finite expectation.

5. (14 points) Given a bounded and continuous function $f(x)$ on $[0, \infty)$, prove that

$$\lim_{n\to\infty} e^{-nx} \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) \frac{(nx)^k}{k!} = f(x) \quad \forall x \ge 0$$

6. (14 points) Let $\{X_n\}$ be an i.i.d. sequence such that $P\{X_1 = e^{-1}\} = P\{X_1 = e\} = 1/2$. Prove that the random sequence

$$\Big(\prod_{k=1}^{n} X_k\Big)^{1/\sqrt{n}} \qquad n = 1, 2, \cdots$$

converges in distribution and determine the limit distribution.

7. (12 points) Let τ be stopping times with respect to filtration $\{\mathcal{F}_n\}$. Show that a random variable Y is $\mathcal{F}_\tau$-measurable if and only if $Y\mathbf{1}\{\tau = n\}$ is $\mathcal{F}_n$-measurable for each n.

8. (12 points) If $\{X_n\}$ is a martingale and it is bounded either from above or below by a constant K, then $\{X_n\}$ is bounded in L^1.

9. (14 points) Let X and Y be integrable random variables with $E(Y|X) = X$ and $E(X|Y) = Y$. Prove that $X = Y$ a.s.

Prelim **PROBABILITY** **August 12, 2019**

Name: ______________________________

1. (12 points) Let A_n be the square $\{(x, y) : |x| \le 1,\ |y| \le 1\}$ pinned at $(0, 0)$ and rotated through the angle $2\pi n\theta$. Give geometric descriptions of $\limsup_n A_n$ and $\liminf_n A_n$ when
   (a) $\theta = 1/8$;
   (b) θ is irrational. [Hint: $2\pi n\theta$ reduced modulo 2π are dense in $[0, 2\pi]$ if θ is irrational.]

2. (12 points) (i) Let A_n be events such that $\mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n\ i.o.) = 0$.
   (ii) Find an example of a sequence A_n to which the result in (i) can be applied but the Borel-Cantelli lemma cannot.

3. (12 points) Let $\{X_n\}$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if $\exists\, c < \infty$ such that $\sum_n \mathbb{P}(X_n > c) < \infty$.

4. (14 points) Let $X_n \ge 0$ be independent random variables. Show that the following are equivalent: (i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s. (ii) $\sum_{n=1}^{\infty} \{\mathbb{P}(X_n > 1) + \mathbb{E}[X_n \mathbf{1}_{)X_n \le 1)}]\} < \infty$.

5. (12 points) Let $S_n = \sum_{i=1}^{n} X_i$, where X_i are i.i.d. with $X_i \ge 0$, $\mathbb{E}X_i = 1$, and $\mathrm{Var}(X_i) = \sigma^2 \in (0, \infty)$. Show that
$$\sqrt{S_n} - \sqrt{n} \ \xrightarrow{d}\ N(0, \sigma^2/4) \quad as\ n \to \infty.$$

6. (14 points) For given $n \ge 1$, let $X_1, \ldots, X_n$ be independent uniform on $[-n, n]$ random variables. Put
$$Y_n = \sum_{i=1}^{n} \frac{\mathrm{sgn}(X_i)}{|X_i|^{\beta}},$$
   where $\beta > 1/2$ is a constant. Show that Y_n converges in distribution as $n \to \infty$ to a random variable with characteristic function $\phi(t) = \exp(-c|t|^{1/\beta})$, where c is a positive constant.

7. (12 points) Let σ and τ be stopping times with respect to filtration $\{\mathcal{F}_n\}$. Show that
   (i) $\sigma \wedge \tau := \min\{\sigma, \tau\}$ is a stopping time and $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$;
   (ii) For any integrable random variable Y
$$\mathbb{E}\{\mathbb{E}[Y \,|\, \mathcal{F}_\tau] \,|\, \mathcal{F}_\sigma\} = \mathbb{E}\{\mathbb{E}[Y \,|\, \mathcal{F}_\sigma] \,|\, \mathcal{F}_\tau\} = \mathbb{E}[Y \,|\, \mathcal{F}_{\sigma\wedge\tau}].$$

8. (10 points) Let $\xi, \xi_1, \xi_2, \cdots$ be random variables with $\xi_n \to \xi$ in L^1. Show for any increasing σ-algebras $\mathcal{F}_n$ that $\mathbb{E}[\xi_n \,|\, \mathcal{F}_n] \to \mathbb{E}[\xi \,|\, \mathcal{F}_\infty]$ in L^1, as $n \to \infty$.

9. (12 points) Let X_n and Y_n be nonnegative integrable random variables adapted to increasing σ-algebras $\mathcal{F}_n$. Suppose $\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] \le X_n + Y_n$, with $\mathbb{E}[\sum_n Y_n] < \infty$. Prove that X_n converges a.s. to a finite limit.
   [Hint: Verify that $W_n = X_n - \sum_{i=1}^{n-1} Y_i$ is a supermartingale.]

Stochastics Preliminary Exam

Friday, January 4, 2019
9:00–13:00

This exam has 2 pages and 9 questions.

Unless otherwise mentioned, the events, sub σ-algebras, and random variables specified in each question are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Question 1. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with distribution μ. For $A \in \mathscr{B}(\mathbb{R})$ with $\mu(A) \in (0,1)$, define

$$\tau = \inf\{k \geq 1; X_k \in A\}.$$

(1) Prove that $\mathbb{P}(\tau < \infty) = 1$.

(2) Prove that X_τ has distribution given by

$$\mathbb{P}(X_\tau \in H) = \frac{\mu(H \cap A)}{\mu(A)}, \quad H \in \mathscr{B}(\mathbb{R}).$$

Question 2. For every $n \geq 1$, let X_n denote the maximum of n independent exponentially distributed random variables $\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_n$, each with mean 1.

(1) Find the explicit form of $\mathbb{P}(X_n < x)$ for all $x \in \mathbb{R}$.

(2) Use (1) to show that

$$X_n - \ln n \xrightarrow[n\to\infty]{\text{(d)}} G,$$

where G has a distribution function given by $\mathbb{P}(G \leq x) = e^{-e^{-x}}$, $x \in \mathbb{R}$.

Question 3. Let $\{X_n\}_{n\geq 1}$ be an arbitrary sequence of random variables. Show that the series $\sum_{n=1}^{\infty} a_n X_n$ converges absolutely a.s. for some constants a_n's.

Question 4. Use an appropriate random variable to construct A_n's such that

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1 \quad \& \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty.$$

Question 5. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that $Y_n = a_n \sum_{j=1}^{n} X_j$ converges in probability to a random variable Y_∞ for some constants $a_n \to 0$. Show that Y_∞ is almost surely equal to a constant.

Question 6. Given any $\lambda \in (0,\infty)$, consider a random variable X_λ taking values in $\mathbb{Z}_+$ with

$$\mathbb{P}(X_\lambda = k) \equiv \frac{e^{-\lambda}\lambda^k}{k!}.$$

(1) Show that $\mathbb{E}[e^{i\theta X_\lambda}] = e^{\lambda(e^{i\theta}-1)}$ for all $\theta \in \mathbb{R}$.

(2) Show that $\mathbb{E}[X_\lambda] = \lambda$ and $\mathrm{Var}(X_\lambda) = \lambda$.

(3) Use (1) and (2) to show that

$$\lim_{N\to\infty} e^{-N}\left(1 + N + \frac{N^2}{2} + \cdots + \frac{N^N}{N!}\right) = \frac{1}{2}.$$

Question 7. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that $\mathbb{E}[X_n] = 0$ for all $n \geq 1$,

$$\sum_{n=1}^{\infty} \mathbb{E}\left[|X_n|\mathbb{1}_{\{|X_n|>1\}}\right] < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{E}\left[X_n^2\mathbb{1}_{\{|X_n|\leq 1\}}\right] < \infty.$$

Prove that $\sum_{n=1}^{\infty} X_n$ converges a.s.

Question 8. Let $\{X_n\}_{n\geq 0}$ be a nonnegative $(\mathscr{F}_n)$-martingale and set

$$\tau = \inf\{n \geq 0; X_n = 0\}.$$

Show that, for every $k \geq 0$, $X_{\tau+k} = 0$ a.s. on $\{\tau < \infty\}$.

Question 9. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_1|^p < \infty$ for some $p \in (0,\infty)$. Define

$$Y_n = \frac{X_n}{n^{1/p}}\mathbb{1}_{\{|X_n|\leq n^{1/p}\}}, \quad \forall\, n \geq 1.$$

Show that for all $\alpha \in (p,\infty)$,

$$\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|^\alpha] \leq \frac{\alpha}{\alpha - p}(\mathbb{E}[|X_1|^p] + 1).$$

Hint: Define $E_j = \{(j-1)^{1/p} < |X_1| \leq j^{1/p}\}$ and show that

$$\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|^\alpha] \leq \sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{-\alpha/p+1}\right)\int_{E_j}(|X_1|^p + 1)d\mathbb{P}.$$

End of questions

Stochastics Preliminary Exam

Friday, August 17, 2018
9:00–13:00

This exam has 2 pages and 9 questions.

Unless otherwise mentioned, the events, sub σ-algebras, and random variables specified in each question are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Question 1. Show that for any sequence of random variables $\{X_n\}$, there exists a sequence of strictly positive constants $\{a_n\}$ such that $a_n X_n$ converges to zero in probability.

Question 2. Let $\{X_n\}$ be a sequence of independent Gaussian random variables with $\mathbb{E}[X_n] = 0$ for all n. Find the probability of the following event:

$$\limsup_{n\to\infty} \{X_n X_{n+1} > 0\}.$$

Question 3. Let $\{A_n\}$ be a sequence of events such that $\lim_{n\to\infty} \mathbb{P}(A_n)$ exists. Prove that the following two properties of $\{A_n\}$ are equivalent:

(1)
$$\lim_{n\to\infty} \mathbb{P}(A_n \cap E) = \lim_{n\to\infty} \mathbb{P}(A_n) \cdot \mathbb{P}(E), \quad \forall\, E \in \mathscr{F}.$$

(2)
$$\lim_{n\to\infty} \int_{A_n} X d\mathbb{P} = \lim_{n\to\infty} \mathbb{P}(A_n) \cdot \int_{\Omega} X d\mathbb{P}, \quad \forall\, X \in L_1(\mathbb{P}).$$

Question 4. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. nonnegative random variables such that for some $p_0 \in (0, \infty)$,

$$\mathbb{E}[X_1^p] < \infty,\ \forall\, p \in (0, p_0] \quad \& \quad \mathbb{E}[X_1^p] = \infty,\ \forall\, p \in (p_0, \infty).$$

Prove that, for p ranging over $(0, \infty)$, we have

$$\limsup_{n\to\infty} \frac{X_n}{n^{1/p}} \begin{cases} = \infty \text{ a.s.} & \forall\, p \in (p_0, \infty); \\ < \infty \text{ a.s.} & \forall\, p \in (0, p_0]. \end{cases}$$

Question 5. Utilize series of the form $\sum_n 1/n^p$ to construct independent, nonnegative random variables X_n such that $\sum_n X_n$ converges a.s. but $\sum_n \mathbb{E}[X_n]$ diverges.

Question 6. Let $\mathscr{P}_0$ denote the set of probability measures μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and define a subset $\mathscr{P}_1$ of $\mathscr{P}_0$ by

$$\mathscr{P}_1 = \left\{ \mu \in \mathscr{P}_0 \,\middle|\, \int_{\mathbb{R}} x\mu(dx) = 0 \;\; \& \;\; \int_{\mathbb{R}} x^2 \mu(dx) = 1 \right\}.$$

Define a map $T : \mathscr{P}_1 \to \mathscr{P}_0$ by

$$T\mu(\Gamma) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\Gamma} \left(\frac{x+y}{\sqrt{2}} \right) \mu(dx)\mu(dy), \quad \Gamma \in \mathscr{B}(\mathbb{R}).$$

(1) Give a probabilistic interpretation to $T\mu$ by explaining how it can be identified as the distribution of a function of suitable random variables identically distributed as μ.

(2) Show that $T\mathscr{P}_1 \subseteq \mathscr{P}_1$.

(3) Prove by the central limit theorem that the following fixed point equation admits a unique solution:

$$\mu = T\mu = \lim_{n\to\infty} T^n \mu$$

and the unique solution is given by $\mathcal{N}(0, 1)$.

Question 7. Let $\{X_n\}$ be an $(\mathscr{F}_n)$-martingale such that

$$\sup_{\omega \in \Omega} |X_1(\omega)| \le K \quad \& \quad \sup_{n \in \mathbb{N},\, \omega \in \Omega} |X_n(\omega) - X_{n-1}(\omega)| \le K$$

for some finite constant K.

(1) Prove that X_τ is integrable for any $(\mathscr{F}_n)$-stopping time τ with $\mathbb{E}[\tau] < \infty$.

(2) Use (1) to show that $\mathbb{E}[X_\tau] = \mathbb{E}[X_1]$ for all stopping times with the same property.

Question 8. Let $\{Y_n\}$ be a sequence of i.i.d. random variables such that Y_n takes values in $\{1/2, 1, 3/2\}$ with probability $1/3$ each. Define $X_n = \prod_{i=1}^n Y_i$. Prove the following properties:

(1) $\{X_n\}$ is a martingale.

(2) $\lim_{n\to\infty} X_n = 0$ a.s. and then explain why $\mathbb{E}\left[\prod_{i=1}^\infty Y_i\right] < \prod_{i=1}^\infty \mathbb{E}[Y_i]$.

Question 9. Let $X \ge 0$ be a random variable with distribution function $F(t)$ such that $F(t) < 1$ for all $t \in \mathbb{R}$ and, for some $\eta \in (1, \infty)$,

$$\lim_{t\to\infty} \frac{1 - F(\eta t)}{1 - F(t)} = 0.$$

Show that $\mathbb{E}[X^m] < \infty$ for any $m \in (0, \infty)$.

End of questions

Stochastics Preliminary Exam

Friday, January 6, 2017

09:00-13:00

Name: ______________________________

Instructions:

1. Answer all questions and **show all work**. You will **not** receive credit if you do not justify your answers where it is necessary.
2. There are 9 questions worth a total of 100 points.
3. Remember to **sign** the Honor Pledge.
4. **Before** submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.
5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ______________________________

Question	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
TOTAL	

1. (12 points) Answer the following questions:

   a. Let $\{X_n\}_{n\geq 1}$ be independent $\mathcal{N}(0,1)$ random variables. Show that $\limsup \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}$ a.s. (Hint: $1-\Phi(x) \sim x^{-1}\phi(x)$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$, and $\phi(x)$ the corresponding density.)

   b. Show that
   $$\mathbb{P}(X_n > a_n \text{ i.o. }) = \begin{cases} 0, & \text{if } \sum \mathbb{P}(X_1 > a_n) < \infty \\ 1, & \text{if } \sum \mathbb{P}(X_1 > a_n) = \infty \end{cases}$$

2. (10 points) Let X and Y be independent variables following the exponential distribution with parameters λ and μ respectively. Let $U = \min\{X,Y\}$, $V = max\{X,Y\}$, and $W = U - V$. Show that U and W are independent.

3. (12 points) Let $X_1, X_2, \ldots, X_n$ be independent standard normal random variables. Find the distribution of the following random variables:

   (a) X_1^2

   (b) $\sum_{i=1}^{n} X_i^2$

4. (12 points) Answer the following questions:

   a. State (with details) one theorem which establishes the continuity of the expectation, $\mathbb{E}$.

   b. Let X_n be a sequence of random variables satisfying $X_n \leq Y$ a.s. for some Y with $\mathbb{E}|Y| \leq \infty$. Show that
   $$\mathbb{E}(\limsup_{n\to\infty} X_n) \geq \limsup_{n\to\infty} \mathbb{E}X_n.$$

   c. Let $f_n(x) = \frac{\cos(x/n)}{x^2}$, for $x \geq 1$. Show that $\int_1^\infty f_n(x)dx \to 1$ as $n \to \infty$.

5. (10 points) Let $X_1, X_2, \cdots, X_n$ be i.i.d. whose common characteristic function ϕ satisfies $\phi'(0) = i\mu$. Show that $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathbb{P}} \mu$ as $n \to \infty$.

6. (10 points) Let $\{X_j\}_{j\geq 1}$ be i.i.d. $\mathcal{N}(1,3)$ random variables. Show that
   $$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

7. (12 points) If τ_1 and τ_2 are stopping times with respect to filtration $\mathcal{F}$, show that $\tau_1 + \tau_2$, $\max\{\tau_1, \tau_2\}$ and $\min\{\tau_1, \tau_2\}$ are stopping times also.

8. (12 points) Let Y_n be a submartingale and let S and T be stopping times satisfying $0 \leq S \leq T \leq N$ for some deterministic N. Show that $\mathbb{E}(Y_0) \leq \mathbb{E}(Y_S) \leq \mathbb{E}(Y_T) \leq \mathbb{E}(Y_N)$.

9. (10 points) Let $\{X_r, r \geq 1\}$ be independent Poisson variables with respective parameters $\{\lambda_r, r \geq 1\}$. Show that $\sum_{r=1}^{\infty} X_r$ converges or diverges almost surely according as $\sum_{r=1}^{\infty} \lambda_r$ converges or diverges.

End of exam.

Stochastics Preliminary Exam

Monday, August 8, 2016

09:00-13:00

Name: ______________________________

Instructions:

1. Answer all questions and **show all work**. You will **not** receive credit if you do not justify your answers where it is necessary.
2. There are 10 questions worth a total of 100 points.
3. Remember to **sign** the Honor Pledge.
4. **Before** submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.
5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ______________________________

Question	Points
1	
2	
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9	
10	
TOTAL	

1. (10 points) If $X_1, X_2, \ldots, X_n$ a sequence of independent identically distributed random variables with pdf f, find the probability density function of the order statistics, $X_{(1)} = min(X_1, X_2, \ldots, X_n)$, and $X_{(n)} = max(X_1, X_2, \ldots, X_n)$, respectively.

2. (10 points) Let $X_r, r \geq 1$ be independent, non-negative and identically distributed with infinite mean. Show that $\limsup_{r\to\infty} \frac{X_r}{r} = \infty$ a.s.

3. (10 points) Answer the following questions:
   (a) Let $X_1, X_2, \ldots, X_n$ be independent exponential variables with parameter λ. Show that that $S_n = \sum_{i=1}^n X_i$ has the $Gamma(n, \lambda)$ distribution.
   (b) If X, Y are two independent random variables distributed according to $Gamma(m, \lambda)$ and $Gamma(n, \lambda)$, respectively, then compute the distribution of $X + Y$.

4. (10 points) Let $X_1, X_2, \ldots$ be independent $N(0,1)$ variables. Find the characteristic functions of the following random variables:
   (a) X_1^2
   (b) $\sum_{i=1}^n X_i^2$

5. (10 points) Provide a (counter)example to each of the following claim.
   (a) There exist sequences of random variables which converge a.s. but not in mean.
   (b) There exist sequences of random variables which converge in mean but not a.s.
   (c) If a sequence converges in probability then it does not necessarily converge in mean.
   (d) If $r > s \geq 1$, and a sequence converges in L^s, then it does not necessarily converge in L^r.

6. (10 points) Let $X_1, X_2, \ldots$ be a sequence of independent non-negative random variables and let $N(t) = \max\{n : \sum_{i=1}^n X_i \leq t\}$. Show that $N(t) + 1$ is a stopping time with respect to a suitable filtration to be specified.

7. (10 points) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common density function f. Suppose that it is known that $f(\cdot)$ is either $p(\cdot)$ or $q(\cdot)$, where p, q are given (different) densities. The statistical problem is to decide which of the two is the true. For this we consider the likelihood ratio

$$Y_n = \frac{p(X_1)p(X_2)...p(X_n)}{q(X_1)q(X_2)...q(X_n)},$$

   and we adopt the strategy that $f = p$ if $Y_n \geq a$ or $f = q$ otherwise. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ the filtration generated by $X_1, X_2, \ldots, X_n$. Is the likelihood ratio, Y_n, a martingale? If, yes, under what constraint?

8. (10 points) Let X_n, Y_n some random variables. Answer the following questions:
   (a) Suppose that X_n converges in distribution to X and Y_n in probability to some constant c. Show that the product X_nY_n converges in distribution to cX
   (b) Suppose that X_n converges in distribution to 0 and Y_n in probability to Y. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x, and $g(x, y)$ is continues at $x = 0$ for all y. Show that $g(X_n, Y_n)$ converges in probability to $g(0, Y)$.

Continued...

9. (10 points) Let $X_r, 1 \leq r \leq n$ be independent and identically distributed with mean μ and finite variance σ^2. Let $\bar{X} = \frac{1}{n}\sum_{r=1}^{n} X_r$. Show that

$$\frac{\sum_{r=1}^{n}(X_r - \mu)}{\sqrt{\sum_{r=1}^{n}(X_r - \bar{X})^2}}$$

converges in distribution to $N(0,1)$.

10. (10 points) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with

$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n} \end{cases}$$

Let $Y_1 = X_1$ and for $n \geq 2$

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1} = 0 \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0 \end{cases}$$

Show that

(a) Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$

(b) Y_n does not converge almost surely. Does it converge in any way?

(c) The martingale convergence theorem does not apply.

End of exam.

University of Tennessee, Knoxville

Stochastics Preliminary Exam

Friday, January 8, 2016

Instructions:

- There are a total of nine (9) problems. An answer without an explanation may receive no credit.
- Throughout the exam, $(\Omega, \mathcal{F}, P)$ is a fixed probability space and given random variables are assumed to be defined on this probability space and take values in $\mathbb{R}$. The expression $\mathbf{1}_A$ denotes the indicator function of a set A.

1. (10 points) Let X be a random variable and let $\sigma(X)$ denote the σ-algebra generated by X. Show that a map $Y : \Omega \to \mathbb{R}$ is $\sigma(X)$-measurable if and only if there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

2. (12 points) Let X and Y be independent Poisson random variables with respective parameters λ and μ.

(a) Show that $X + Y$ has a Poisson distribution with parameter $\lambda + \mu$.

(b) According to Problem 1, the conditional expectation $E[X|X+Y]$ can be written as a function of $X + Y$; i.e. there exists $f : \mathbb{R} \to \mathbb{R}$ such that

$$E[X|X+Y] = f(X+Y).$$

Specify the function f.

3. (10 points) Let $(X_n)_{n\geq 1}$ be a sequence of random variables and X be a random variable. Show that $X_n \to X$ in probability as $n \to \infty$ if and only if

$$\lim_{n\to\infty} E[|X_n - X| \wedge 1] = 0,$$

where $a \wedge b := \min\{a, b\}$.

4. (10 points) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $P(X_n = 1) = P(X_n = -1) = 1/2$. What can be concluded about the probability

$$P\left(\sum_{n=1}^{\infty} \frac{X_n}{n} \text{ converges}\right)?$$

5. (10 points) Let X be an integrable random variable. Let Λ be the family of all sub-σ-algebras of $\mathcal{F}$. Let $Y_{\mathcal{G}} := E[X|\mathcal{G}]$ for $\mathcal{G} \in \Lambda$. Show that the family $(Y_{\mathcal{G}})_{\mathcal{G}\in\Lambda}$ is uniformly integrable.

6. (10 points) Using the central limit theorem, prove that

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{a}\sin\frac{a}{\sqrt{n}}\right)^n = e^{-a^2/6},$$

where a is a nonzero real constant.

Hint: Consider an i.i.d. sequence of uniform random variables on $[-1,1]$.

7. (14 points) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common density function $f(x) = e^{-x}\mathbf{1}_{(0,\infty)}(x)$, $x \in \mathbb{R}$. For a fixed $\theta < 1$, let

$$Y_n := (1-\theta)^n e^{\theta S_n}$$

for $n \geq 1$, where $S_n = \sum_{k=1}^n X_k$.

(a) Show that there exists $Y_\infty \in L^1$ such that $Y_n \to Y_\infty$ a.s.

(b) Suppose further that $\theta \neq 0$. Find the a.s. limit of the sequence $(\frac{1}{n}\log Y_n)$, and deduce that $Y_\infty = 0$ a.s.

8. (10 points) Let $(X_n)_{n\geq 1}$ be a sequence of random variables. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel measurable function such that $f(x) \to \infty$ as $|x| \to \infty$ and $\sup_{n\geq 1} E[f(X_n)] < \infty$. Show that (X_n) is tight.

9. (14 points) Let $(A_n)_{n\geq 1}$ be events in $\mathcal{F}$ satisfying the following two conditions:

- $\displaystyle\liminf_{n\to\infty}\frac{\sum_{j,k=1}^n P(A_j\cap A_k)}{\left(\sum_{j=1}^n P(A_j)\right)^2} = 1.$
- $\displaystyle\sum_{n=1}^\infty P(A_n) = \infty.$

(a) Show that

$$\liminf_{n\to\infty} P\left(\left|\sum_{k=1}^n \mathbf{1}_{A_k} - \sum_{k=1}^n P(A_k)\right| > \frac{1}{2}\sum_{k=1}^n P(A_k)\right) = 0.$$

(b) Show that there is a subsequence $\{n_m\}_{m=1}^\infty$ of $\mathbb{N}$ such that with probability 1,

$$\sum_{k=1}^{n_m}\mathbf{1}_{A_k} \geq \frac{1}{2}\sum_{k=1}^{n_m} P(A_k) \quad \text{for sufficiently large } m.$$

Hint: Consider the sequence $c_n := P\left(\sum_{k=1}^n \mathbf{1}_{A_k} < \frac{1}{2}\sum_{k=1}^n P(A_k)\right)$.

(c) Show that $P(\limsup_{n\to\infty} A_n) = 1$.

— END OF EXAM —

University of Tennessee, Knoxville

Stochastics Preliminary Exam

Monday, August 10, 2015

Instructions:

- There are a total of eight (8) problems. An answer without an explanation may receive no credit.
- Throughout the exam, $(\Omega, \mathcal{F}, P)$ is a fixed probability space and given random variables are assumed to be defined on this probability space and take values in $\mathbb{R}$. The expression $\mathbf{1}_A$ denotes the indicator function of a set A.

1. (16 points)

(a) Suppose that X is an integrable random variable and $\{A_n\}_{n\geq 1}$ is a sequence of events in $\mathcal{F}$ such that $P(A_n) \to 0$ as $n \to \infty$. Show that $E[X\mathbf{1}_{A_n}] \to 0$ as $n \to \infty$.

(b) Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of random variables such that $X_n \to X$ in probability for some random variable X. Suppose further that there exists a random variable $Y \in L^p$ for some $p \geq 1$ such that $|X_n| \leq Y$ a.s. for all $n \geq 1$. Prove that $X_n \to X$ in L^p.

2. (12 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with common density $f(x) = xe^{-x^2/2}\mathbf{1}_{(0,\infty)}(x)$, $x \in \mathbb{R}$. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \quad \text{a.s.}$$

3. (14 points)

(a) Show that a random variable X is symmetric (i.e. $X =^{\mathrm{d}} -X$) if and only if its characteristic function is real-valued.

(b) Show that if X and Y are i.i.d. random variables, then $X - Y$ is symmetric.

4. (10 points) Let $\varphi(t)$, $t \in \mathbb{R}$, be the characteristic function of a random variable X. Show that if $\varphi(t) = 1$ in a neighborhood of 0, then $X = 0$ a.s.

Hint: Show that $1 - \mathrm{Re}[\varphi(2t)] \leq 4(1 - \mathrm{Re}[\varphi(t)])$ for $t \in \mathbb{R}$.

5. (10 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with common mean μ and variance $\sigma^2 \in (0, \infty)$. Use Slutsky's theorem to show that

$$\sqrt{n}(e^{S_n/n} - e^{\mu}) \Longrightarrow \sigma e^{\mu} \chi,$$

where $S_n = \sum_{k=1}^{n} X_k$ and $\chi \sim N(0, 1)$.

6. (10 points) Show that if $\{X_n\}_{n\geq 1}$ is a sequence of random variables such that $\sup_{n\geq 1} E[|X_n|^p] < \infty$ for some $p > 0$, then $\{X_n\}_{n\geq 1}$ is tight.

7. (10 points) For a filtration $(\mathcal{F}_n)_{n\geq 0}$ on the probability space $(\Omega, \mathcal{F}, P)$, let $(M_n)_{n\geq 0}$ be an $(\mathcal{F}_n)$-submartingale and let $(H_n)_{n\geq 1}$ be an $(\mathcal{F}_n)$-predictable process such that each H_n is nonnegative and bounded. Show that the stochastic process $(Y_n)_{n\geq 0}$ defined by

$$Y_0(\omega) := 0, \quad Y_n(\omega) := \sum_{k=1}^{n} H_k(\omega)(M_k(\omega) - M_{k-1}(\omega)), \ n \geq 1, \ \omega \in \Omega,$$

is an $(\mathcal{F}_n)$-submartingale.

8. (18 points) Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. nonnegative random variables such that $E[X_0] = 1$ and $P(X_0 = 1) < 1$. Let $Y_n := \Pi_{j=0}^{n} X_j$ for $n \geq 0$.

(a) Show that $Y_n \to Y_\infty$ a.s., where Y_∞ is a finite random variable.

(b) For fixed $\epsilon, \delta > 0$ and $n \geq 0$, show that

$$P(|Y_{n+1} - Y_n| > \epsilon\delta) \geq P(|Y_n| > \epsilon)P(|X_0 - 1| > \delta).$$

(c) Show that $Y_\infty = 0$ a.s.

— END OF EXAM —

Name: ______________________________

1. (14 points) (a) Let A_n be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\mathbb{P}(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mathbb{P}(A_n) \le \limsup_{n\to\infty} \mathbb{P}(A_n) \le \mathbb{P}(\limsup_{n\to\infty} A_n).$$

(b) Let X_n be i.i.d. random variables such that $\mathbb{P}(X_n = 1) = p$, $\mathbb{P}(X_n = 0) = 1-p$, with $p \in (0,1)$. Let s be any m-long sequence of zeros and ones and let $A_n = \{\omega : (X_n(\omega), \dots, X_{n+m}(\omega)) = \mathsf{s}\}$. Show that $A := \{A_n \text{ i.o.}\}$ is a tail event for $\{X_n\}$ and determine $\mathbb{P}(A)$.

2. (10 points) Let X_n be random variables such that for some $a_n \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq a_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

Show that $\sum_n X_n$ converges a.s.

3. (10 points) Let X be a random variable.

(a) X is independent of itself (i.e., X and X are independent) if and only if there is a constant c such that $\mathbb{P}(X = c) = 1$. [Hint: Consider the distribution function of X.]

(b) X is independent of $g(X)$ for some measurable function $g : \mathbb{R} \mapsto \mathbb{R}$ if and only if there is a constant c such that $\mathbb{P}(g(X) = c) = 1$.

4. (12 points) Let $\{X_n\}$ be i.i.d. Normal$(0,1)$ random variables. Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\right) = 1.$$

[Hint: You may use $\int_x^\infty e^{-u^2/2}\,du \sim \frac{1}{x} e^{-x^2/2}$ as $x \to \infty$.]

5. (12 points) Let X_n be random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \Rightarrow X$. Suppose that $\mathbb{P}(X > b) = \delta > 0$. Show that $\mathbb{P}(X_n \ge b \text{ i.o.}) \ge \delta$. [Hint: You may use Problem 1 (a).]

6. (10 points) Let X_j be i.i.d. random variables with the common distribution function F. Define for $x \in \mathbb{R}$

$$Y_j(x) = \mathbf{1}_{\{X_j \le x\}}$$

and

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} Y_j(x).$$

Show that $\forall\, x \in \mathbb{R}$,

(i)

$$\lim_{n\to\infty} F_n(x) = F(x) \quad a.s.$$

and

(ii)

$$\sqrt{n}(F_n(x) - F(x)) \Rightarrow Z$$

where Z is normal with mean zero and variance $\sigma^2(x) = F(x)(1 - F(x))$.

7. (10 points) Let $\{X_n\}$ be i.i.d. random variables with $\mu = E(X_1)$ and $\sigma^2 = \text{Var}(X_1) < \infty$. Put $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$. Let h be a measurable function that is differentiable at μ and $h'(\mu) \neq 0$. Show that
$$\sqrt{n}\sigma^{-1}(h(\overline{X}_n) - h(\mu)) \Rightarrow h'(\mu)Z \quad \text{as } n \to \infty,$$
where Z is a standard normal random variable. [Hint: Consider $\frac{h(\bar{X}_n)-h(\mu)}{\bar{X}_n-\mu}$ as $n \to \infty$.]

8. (10 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let X be an integrable random variable such that $E(X \mid \mathcal{G}) \leq X$ a.s. Show that $X = E(X \mid \mathcal{G})$ a.s.

9. (12 points) A martingale $\{X_n\}$ is bounded in L^2 if $\sup_n \mathbb{E}X_n^2 < \infty$. Show that a martingale $\{X_n\}$ is bounded in L^2 if and only if $\mathbb{E}X_n^2 < \infty$ for each n and
$$\sum_{n=1}^{\infty} \mathbb{E}\{(X_{n+1} - X_n)^2\} < \infty.$$

Prelim Exam | PROBABILITY | August 15, 2014

Name: ______________________________

1. (10 points) Show the following extension of subadditivity: for any events $B_i \subset A_i$
$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) - \mathbb{P}\Big(\bigcup_{i=1}^{n} B_i\Big) \le \sum_{i=1}^{n}\Big(\mathbb{P}(A_i) - \mathbb{P}(B_i)\Big)$$
where $n \le \infty$.

2. (12 points) For any $n \in \mathbb{N}$, let $\{X_{n,k} : 1 \le k \le n\}$ be i.i.d. random variables such that $0 \le X_{n,k} \le C$ (same constant C for all n and k), and let $S_n = \sum_{k=1}^{n} X_{n,k}$. Show that if $\mu_n := \mathbb{E}S_n \to \infty$, then $S_n \xrightarrow{\mathbb{P}} \infty$ (that is, $\forall M > 0$ $\mathbb{P}(S_n > M) \to 1$).
(Hint: Since $\mu_n \to \infty$, it is enough to show that $\mathbb{P}(S_n \in (\frac{\mu_n}{2}, \frac{3\mu_n}{2})) \to 1$.)

3. (12 points) Let $\{X_n\}$ be i.i.d. random variables having a Weibull distribution with density $f(x) = 3x^2 \exp(-x^3)$ for $x \ge 0$ and $f(x) = 0$ otherwise. Show that
$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\sqrt[3]{\log n}} = 1\right) = 1\,.$$

4. (14 points) Let $\{X_j\}$ be i.i.d. random variables and let N be a Poisson(λ) random variable independent of $\{X_j\}$. Let $S_n = \sum_{j=1}^{n} X_j$ and consider S_N.
(a) Find the characteristic function of S_N.
(b) Find $\mathbb{E}S_N$ and $\mathrm{Var}(S_N)$.
(c) Let $\{\epsilon_j\}$ be i.i.d. random variables independent of $\{X_j\}$ and N such that $\mathbb{P}(\epsilon_j = 1) = p$, $\mathbb{P}(\epsilon_j = 0) = 1 - p$. Show that $S'_N := \sum_{j=1}^{N} \epsilon_j X_j$ and $S''_N := \sum_{j=1}^{N}(1 - \epsilon_j)X_j$ are independent.

5. (10 points) Let X_n be independent random variables such that $X_n \xrightarrow{\mathbb{P}} X$.
(a) Show that there is a constant c such that $\mathbb{P}(X = c) = 1$.
(b) If X_n are iid and not equal to a constant, then no such X exists.

6. (10 points) Let X_j be i.i.d. and $S_n = \sum_{j=1}^{n} X_j$. Show that if $\dfrac{S_n}{n} \to 0$ a.s., then $\mathbb{E}|X_j| < \infty$ and also $\mathbb{E}X_j = 0$.

7. (12 points) (a) Suppose X and Y are i.i.d. $N(0,1)$. Show $\frac{X+Y}{\sqrt{2}} \stackrel{d}{=} X \stackrel{d}{=} Y$.
(b) Conversely: Suppose X and Y are i.i.d. with mean zero and variance 1, and suppose further that
$$\frac{X+Y}{\sqrt{2}} \stackrel{d}{=} X \stackrel{d}{=} Y\,.$$
Show that both X and Y have a $N(0,1)$ distribution. (Use the Central Limit Theorem.)

8. (10 points) Let $\mathcal{A}_1 \subset \mathcal{A}_2$ be sub-σ-algebras of $\mathcal{F}$. Show that for any random variable $X \in L^2$
$$\mathbb{E}((X - \mathbb{E}(X|\mathcal{A}_2))^2) \le \mathbb{E}((X - \mathbb{E}(X|\mathcal{A}_1))^2)\,.$$
Discuss extremal cases of this inequality: $\mathcal{A}_1 = \{\emptyset, \Omega\}$ and $\mathcal{A}_2 = \mathcal{F}$.

9. (10 points) Let $\{X_n\}$ be a martingale such that $Y_n = \dfrac{X_{n+1}}{X_n} \in L^1$. Show that $\mathbb{E}Y_n = 1$ and, if $Y_n \in L^2$, then $\mathrm{Cov}(Y_n, Y_{n+1}) = 0$.

Stochastics Preliminary Exam

Friday, January 3, 2014

09:00-13:00

Name: ______________________________

Instructions:

1. Answer all questions and **show all work**. You will **not** receive credit if you do not justify your answers where it is necessary.
2. There are 7 questions worth a total of 100 points on 2 numbered pages. Please check the pages before start working.
3. Remember to **sign** the Honor Pledge.
4. **Before** submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.
5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ______________________________

1. Let $A_r, r \geq 1$ be events such that $\mathbb{P}(A_r) = 1$ for each r. Show that $\mathbb{P}(\cap_{r=1}^{\infty} A_r) = 1$.

2. Consider two random variables X_t and Z_t whose distributions change with time t. One may observe Z_t but not X_t and the conditional distribution $Z_t|X_t = x_t \sim p(Z_t|x_t)$ is well-defined, where x_t is the state of X_t at time t. Denote $x_{0:t} = \{x_0, \cdots, x_t\}$ the whole history of states where the random variable X visited from time 0 to time t. Furthermore, consider that the transition density $p(X_t|X_{0:t-1} = x_{0:t-1}) = p(X_t|X_{t-1} = x_{t-1})$ is known.
   (a) Employing Bayes theorem calculate the posterior distribution $p(X_t|Z_{0:t} = z_{0:t})$.
   (b) If $Z_t|X_t = x_t \sim \mathcal{N}(x_t, 1)$ and $X_t|X_{t-1} = x_{t-1} \sim N(x_{t-1}, 1)$ then give a closed form of the posterior distribution $p(X_t|Z_{0:t} = z_{0:t})$.

3. (a) Consider a sequence of random variables $\{X_n\}$ which converge to some limit X. Define the a.s., in probability, in distribution and in L^p convergence and state (without any proof) any relationship among these types of convergence.
   (b) Suppose $X_n \xrightarrow{L^2} X$. Show that $\mathbb{V}ar(X_n) \to \mathbb{V}ar(X)$ as $n \to \infty$.

4. (a) Let $\{X_n\}_{n\geq 1}$ be independent and identically distributed (i.i.d.) where $X_1 \sim Exp(1)$. Show that $\limsup_{n\to\infty} \frac{X_n}{log n} = 1$ a.s.
   (b) Let $\{X_n\}_{n\geq 1}$ be i.i.d., non-negative with infinite mean. Show that $\limsup_{n\to\infty} \frac{X_n}{n} = \infty$ a.s.

5. Let Y_n be a submartingale and let S and T be stopping times satisfying $0 \leq S \leq T \leq N$ for some deterministic N. Show that $\mathbb{E}(Y_0) \leq \mathbb{E}(Y_S) \leq \mathbb{E}(Y_T) \leq \mathbb{E}(Y_N)$.

6. Let Y be a submartingale, u a convex non-decreasing function mapping $\mathbb{R}$ to $\mathbb{R}$. Show that $\{u(Y_n) : n \geq 0\}$ is a submartinagle provided that $\mathbb{E}u(Y_n)^+ < \infty$ for all n.

7. (a) Let $X_1, X_2, \cdots, X_n$ be independent random variables with characteristic functions $\phi_1, \phi_2, \cdots, \phi_n$. Match the following characteristic functions (dots) with the random variables (numbers). Place your selection in the brackets.
   - $\prod_{i=1}^{n} \phi_i(t)$: [ ]
   - $|\phi_1(t)|^2$: [ ]
   - $\sum_{j=1}^{n} p_j \phi_j(t)$, where $p_j \geq 0$ and $\sum_{i=1}^{n} p_j = 1$: [ ]
   - $(2 - \phi_1(t))^{-1}$: [ ]
   - $\int_0^{\infty} \phi_1(ut) e^{-u} du$: [ ]

   1. $\sum_{i=1}^{N} X_i$, where N is a random variable with $\mathbb{P}(N = j) = p_j$ for $1 \leq j \leq n$, independent of $X_1, X_2, \cdots, X_n$
   2. $\sum_{i=1}^{n} X_i$
   3. $X_1 - X_1'$, where X_1, X_1' are i.i.d.
   4. YX_1, where Y is independent of X_1 and follows Exp(1).
   5. $\sum_{j=1}^{M} Z_j$, where $Z_1, Z_2, \cdots$ are independent and distributed as X_1 and M is independent of the Z_j with $\mathbb{P}(M = m) = \frac{1}{2^{m+1}}$, for $m \geq 0$

   (b) Is the function $\phi(t) = (1 + t^4)^{-1}$ a characteristic function?

End of exam.

Stochastics Preliminary Exam

Monday, August 12, 2013

09:00-13:00

Name: ______________________________

Instructions:

1. Answer all questions and **show all work**. You will **not** receive credit if you do not justify your answers where it is necessary.
2. There are 10 questions worth a total of 100 points.
3. Remember to **sign** the Honor Pledge.
4. **Before** submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.
5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ______________________________

1. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ satisfies $\mathbb{P}(B) > 0$. Let $\mathbb{Q} : \mathcal{F} \to [0,1]$ be defined by $\mathbb{Q}(A) = \mathbb{P}(A|B)$. Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A|C) = \mathbb{P}(A|BC)$.

2. There are 10 coins in a bag. Five of them are normal coins (one head and one tail), one coin has two heads and four coins have two tails. You pull one coin out, look at one of its sides and see that it is a tail. What is the probability that it is a normal coin?

3. Let X be a nonnegative random variable following a distribution F. Show that $\mathbb{E}X = \int_0^\infty (1 - F(x))dx$.

4. Suppose that the real-valued random variables ξ, η are independent, that ξ has a bounded density $p(x)$ (for $x \in \mathbb{R}$, with respect to Lebesgue measure), and that η is integer valued.

   i. Prove that $\zeta = \xi + \eta$ has a density.

   ii. Calculate the density of ζ in the case where $\xi \sim$ Uniform[0,1] and $\eta \sim$ Poisson(1).

5. Let $X_1, X_2, \cdots$ be independent random variables with zero means and $S_n = X_1 + \cdots + X_n$. Let $M_n = \max_{1 \le k \le n} |S_k|$.

   i. Show that $\mathbb{E}(S_n^2 I_{A_k}) > c^2 \mathbb{P}(A_k)$, where $A_k = \{M_{k-1} \le c < M_k\}$ and $c > 0$.

   ii. Deduce Kolmogorov's inequality: $\mathbb{P}\Big(M_n > c\Big) \le \frac{\mathbb{E}(S_n^2)}{c^2}$, $c > 0$.

6. Let $X_t, t > 0$ be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assuming that $E|X_t|^2 = E|X|^2 < \infty$ for all t, prove that $\mathbb{P}(\lim_{t\to 0} X_t = X) = 1$ implies $\lim_{t \to 0} E|X_t - X|^2 = 0$, i.e. under the above assumptions, almost sure convergence implies convergence in mean square.

7. Let $\{X_n\}$ be a sequence of independent random variables which converges in probability to the limit X. Show that X is almost surely constant.

8. Let X_n be independent and identically distributed random variables with $\mathbb{P}(X_n = 1) = \frac{1}{2}$ and $\mathbb{P}(X_n = -1) = \frac{1}{2}$. Show that if $S_n = \sum_{j=1}^n X_j$ then

   i. $\frac{1}{n} S_n \xrightarrow{\mathbb{P}} 0$

   ii. $\frac{1}{n^2} S_{n^2} \xrightarrow{a.s.} 0$

9. Let $X_1, X_2, \cdots$ be independent identically distributed random variables such that $\mathbb{E}X_n = 0$ and $|X_n| \le 1$ a.s. Define $S_n = \sum_{k=1}^n X_k$. Find a number c such that $S_n^2 - cn$ is a martingale and justify the martingale property.

10. Let $(Y, \mathcal{F})$ be a martingale and suppose that there exists a sequence $K_1, K_2, \cdots$ of real numbers such that $\mathbb{P}(|Y_n - Y_{n-1}| \le K_n) = 1$ for all n. Show that $\mathbb{P}(|Y_n - Y_0| \ge x) \le 2\exp\Big(-\frac{\frac{1}{2}x^2}{\sum_{i=1}^n K_i^2}\Big)$, $x > 0$. (*Hint: Consider a random variable D with 0 mean and $\mathbb{P}(|D| \le 1) = 1$. Use the inequality $e^{\psi d} \le \frac{1}{2}(1-d)e^{-\psi} + \frac{1}{2}(1+d)e^{\psi}$, if $|d| \le 1$, $\psi > 0$ to bound $\mathbb{E}(e^{\psi D})$. Furthermore, find a pertinent D based on the hypothesis of the problem.*)

End of exam.

Prelim Exam for Stochastics
January 4, 2013

Name ____________________
ID number ____________________

1. (11 points) Let X be a random variable with $\mathbb{E}X = 0$ and finite variance σ^2.

a) For any t, show that

$$\mathbb{E}\frac{X^2}{X^2+t^2} \le \frac{\sigma^2}{\sigma^2+t^2}.$$

b) Use last inequality to prove that for $t > 0$,

$$\mathbb{P}(X \le t) \ge \frac{t^2-\sigma^2}{t^2+\sigma^2}.$$

2. (12 points) A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses. Show that $p_0 = 1$ and

$$p_n = p + (1-2p)p_{n-1} \qquad \text{if } n \ge 1.$$

Solve this difference equation.

3. (12 points) A telephone sales company attempts repeatedly to sell new kitchens to each of the N families in a village. Family i agrees to buy a new kitchen after it has been solicited K_i times, where the K_i are i.i.d. random variables with probability mass function

$$f(n) = \mathbb{P}(K_i = n), \qquad n = 1, 2, \cdots.$$

Let X_n be the number of kitchens sold at the nth round of solicitations, so that

$$X_n = \sum_{i=1}^{N} 1_{K_i=n}.$$

Suppose that N is a Poisson random variable (independent of K_i's) with parameter ν.

a) Show that the X_n are independent random variables, X_r having the Poisson distribution with parameter $\nu f(r)$. (Hint: Use characteristic function).

b) Let

$$T = \inf\{n : X_n = 0\} \text{ and } S = X_1 + X_2 + \cdots + X_T.$$

Show that

$$\mathbb{E}(S) = \nu \mathbb{E} F(T),$$

where

$$F(k) = \sum_{j=1}^{k} f(j).$$

4. (12 points) Let X_n, $n = 1, 2, \cdots$ be i.i.d. random variable with common probability density function $f(x) = e^{-x}$, $x > 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1, \qquad a.s.$$

Hint: Prove that

$$P\left(\limsup_n A_n^{1-\epsilon}\right) = 1 \text{ and } P\left(\limsup_n A_n^{1+\epsilon}\right) = 0,$$

where

$$A_n^c = \left\{\frac{X_n}{\log n} > c\right\}.$$

5. (10 points) Let f be an increasing function on interval $[a, b]$. Use probability method to show that

$$\int_a^b 2x f(x) dx \geq (a+b) \int_a^b f(x) dx.$$

6. (12 points) a) Let X_λ be a Poisson random measure with parameter λ. Prove that $\frac{X_\lambda - \lambda}{\sqrt{\lambda}}$ converges in distribution. Identify the limiting distribution.

b) Prove that

$$\lim_{n\to\infty} e^{-nt} \sum_{k=1}^{n-1} \frac{(nt)^k}{k!} = \begin{cases} 1 & \text{if } 0 < t < 1, \\ \frac{1}{2} & \text{if } t = 1, \\ 0 & \text{if } t > 1. \end{cases}$$

(Hint: Use part a).)

7. (11 points) For any nonnegative random variable X, prove that

$$\mathbb{E}X \leq \sum_{n=0}^{\infty} P(X > n) \leq \mathbb{E}X + 1.$$

8. (10 points) Let X and Y be two integrable random variables. Prove that for any bounded Borel function f,

$$\mathbb{E}\left((X - f(Y))^2\right) \geq \mathbb{E}\left((X - \mathbb{E}(X|Y))^2\right).$$

9. (10 points) Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a sequence of random variables. Define

$$\mathcal{F}_n = \sigma(X_1, X_2, \cdots, X_n).$$

Suppose that for any bounded stopping times $\sigma \leq \tau$, we have

$$\mathbb{E}X_\sigma \leq \mathbb{E}X_\tau, \qquad a.s.$$

Prove that $(X_n, \mathcal{F}_n)$ is a submartingale.

Probability Prelim

August 8, 2011

1. Let $\{X_n\}_{k\geq 1}$ be an independent sequence of random variables such that

$$P\{X_n = 0\} = 1 - n^{-p} \quad \text{and} \quad P\{X_n = n^p\} = n^{-p} \qquad n = 1, 2, \cdots,$$

where $p > 0$.

(a). Determine the values of p that make the limit of $\{X_n\}_{k\geq 1}$ exist almost surely, and find the strong limit $\lim_{n\to\infty} X_n$ when it exists.

(b) Directly exam "true or false" for the statements

$$E\Big(\lim_{n\to\infty} X_n\Big) = \lim_{n\to\infty} EX_n \quad \text{and} \quad E\Big(\liminf_{n\to\infty} X_n\Big) < \liminf_{n\to\infty} EX_n$$

whenever the problem is well-posted.

2. Let $h(x)$ be a bounded, strictly increasing and continuous function on $[0, \infty)$ such that $h(0) = 0$. Prove that for any random variables X_n and X, $X_n \xrightarrow{p} X$ if and only if

$$\lim_{n\to\infty} Eh(|X_n - X|) = 0.$$

3. Let $\{X_k\}_{k\geq 1}$ be an i.i.d. sequence and let $\{\xi_n\}_{n\geq 1}$ be a sequence of Poisson random variables with $E\xi_n = n$ ($n = 1, 2, \cdots$). Assume independence between $\{X_k\}_{k\geq 1}$ and $\{\xi_n\}_{n\geq 1}$.

(a). Compute the characteristic function of the variable

$$Z_n = \sum_{k=1}^{\xi_n} X_k$$

(More precisely, represent the characteristic function of Z_n in terms of the characteristic function of X_1).

(b). Assume that $EX_1 = 0$ and $EX_1^2 = 1$. Prove that the random sequence $Z_n/\sqrt{n}$ converges in distribution and identify the limiting distribution.

4. Let $\varphi(x)$ be an non-negative and convex function. Let X and Y be two independent random variables with $EX = 1$. Prove that

$$E\varphi(Y) \leq E\varphi(XY).$$

5. Let $X \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$ and let $\mathcal{G} \subset \mathcal{A}$ be a sub σ-algebra. Assume that $X \stackrel{d}{=} E\big[X|\mathcal{G}\big]$. Prove that $X = E\big[X|\mathcal{G}\big]$ a.s.

6. Let (X, Y) be a 2-dimensional Gaussian random variable.

(a). Prove that there are constants a and b such that

$$E\big[Y|X\big] = aX + b$$

and determine a and b as much as you can. Hint: You may start by solving and justfying the equation

$$\begin{cases} \mathrm{Cov}\Big(Y - (aX + b),\ X\Big) = 0 \\ EY = aEX + b. \end{cases}$$

(b). Prove that the conditional variance defined as

$$\mathrm{Var}(Y|X) = E\Big\{\big(Y - E[Y|X]\big)^2 |X\Big\}$$

is equal to a constant almost surely.

7. Let $\{X_k\}_{k\geq 1}$ be an independent sequence such that $EX_k = 0$ and

$$\sum_{k=1}^{\infty} EX_k^2 < \infty.$$

Prove that the random series

$$\sum_{k=1}^{\infty} X_k$$

converges almost surely.

8. Let $\{X_k\}_{k\geq 1}$ be an i.i.d. sequence with the common distribution

$$P\{X_1 = -1\} = P\{X_1 = 1\} = \frac{1}{2}.$$

The sequence

$$S_n = \sum_{k=1}^{n} X_k \qquad n = 1, 2, \cdots$$

is called simple random walk in literature.

(a). For each integer $a \geq 1$, define the stopping time T_a as

$$T_a = \inf\{k \geq 1;\ \ |S_k| = a\}$$

Prove that T_a and S_{T_a} are independent.

(b). Prove that for any real number θ, the sequence $\{M_n\}_{n\geq 0}$ defined as

$$M_0 = 1 \ \text{ and } \ M_n = \Big(\cosh\theta\Big)^{-n} \exp\big\{\theta S_n\big\} \qquad n = 1, 2, \cdots$$

is a martingale.

(c). Prove that

$$E\Big(\cosh\theta\Big)^{-T_a} = \Big(\cosh a\theta\Big)^{-1}.$$

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Name: ______________________________

1. (10 points) Let X, Y, Z be real random variables such that $X \leq Y \leq Z$ and $X, Z \in L^1$. Prove that $Y \in L^1$.

2. (10 points) Show that if random variables X and $-X$ have the same distributions, then the characteristic function of X is real valued.

3. (15 points) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that X_n has density

$$f(x) = \begin{cases} xe^{-x^2/2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Prove that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \quad a.s.$$

4. (14 points) Let X_n be a Poisson random variable with parameter $\lambda_n > 0$, $n = 1, 2, \ldots$. Prove that

   (a) the sequence $(X_n)_{n\geq 1}$ converges in distribution if and only if $(\lambda_n)_{n\geq 1}$ converges.

   (b) Deduce from (a) that if $X_n \xrightarrow{\mathcal{D}} X$ then X is either a Poisson random variable or $X = 0$ a.s.

5. (12 points) Let $S_n = Z_1^2 + \cdots + Z_n^2$, where $(Z_n)_{n\geq 1}$ is a sequence of i.i.d. standard normal random variables.

   (a) Show that $\sqrt{S_n} - \sqrt{n} \xrightarrow{\mathcal{D}} Y$, where Y is NORMAL(0, 1/2).

   (b) Deduce from (a) that $\mathbb{P}(S_n \leq x) \approx \Phi(\sqrt{2x} - \sqrt{2n})$ for every $x \geq 0$ when n is large.

6. (12 points) Let $S_n = X_1 + \cdots + X_n$ be the partial sum of i.i.d. random variables (X_k) with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$. Prove that

   (a) S_n^2 is a submartingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n = 1, 2, \ldots$.

   (b) for any $\alpha > 1/2$,

$$\frac{1}{n^\alpha} \max_{1\leq k\leq n} |S_k| \xrightarrow{\mathbb{P}} 0.$$