Analysis Diagnostic Exam August 7, 2024

NAME: ______________________________

#1.)	____/10	#2.)	____/10	#3.)	____/10	#4.)	____/10	#5.)	____/10
#6.)	____/10	#7.)	____/10	#8.)	____/10			Total:	______/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (10 points) Suppose A, B, and C are sets. If $A \subseteq C$, prove that

$$A \cap B = (B \cap C) \setminus (C \setminus A).$$

2.) (10 points) Suppose A, B, X and Y are non-empty sets, and $f : A \to X$ and $g : B \to Y$ are functions. Define a function $h : A \times B \to X \times Y$ by

$$h(a, b) = (f(a), g(b)),$$

for $a \in A$ and $b \in B$. Let $U \subseteq X$ and $V \subseteq Y$ be non-empty sets. Prove that

$$h^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V).$$

3.) (10 points) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Suppose $\{x_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{x_n\}$. Prove that

$$\lim_{k\to\infty} x_{n_k} \leq \limsup x_n.$$

Here $\limsup x_n$ is defined to be $\lim_{n\to\infty} \sup\{x_k : k \geq n\}$. Don't use any other characterization of $\limsup x_n$ without proof.

4.) (10 points) Using only the $\epsilon - N$ definition of the limit of a sequence (that is, not using any limit theorems without proof), prove that

$$\lim_{n\to\infty} \frac{6n^3 + n - 4}{2n^3 + 50} = 3.$$

5.) (10 points) (a) (7 points) Suppose $A_\lambda \subseteq \mathbb{R}$ for each $\lambda \in \Lambda$, where Λ is any set. Prove that

$$\overline{\cap_{\lambda\in\Lambda} A_\lambda} \subseteq \cap_{\lambda\in\Lambda} \overline{A_\lambda}.$$

On the left side, $\overline{\cap_{\lambda\in\Lambda} A_\lambda}$ means the closure of the intersection.

(b) (3 points) Give an example of sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

6.) (10 points) Let $A = [-1, 0) \cup (0, 1]$. Find a collection of subsets $\{O_n\}_{n=1}^{\infty}$ of $\mathbb{R}$ such that:

(a) $\{O_n\}_{n=1}^{\infty}$ is an open cover of A,

and

(b) no finite subcollection of $\{O_n\}_{n=1}^{\infty}$ covers A.

You should prove that the sets $\{O_n\}_{n=1}^{\infty}$, that you define, satisfy (a) and (b).

7.) (10 points) Suppose $f : (-1, 1) \to \mathbb{R}$ is a function. Suppose that f is continuous at $x = 0$.

(a) (5 points) Define $g : (-1, 1) \to \mathbb{R}$ by

$$g(x) = xf(x).$$

Prove that g is differentiable at $x = 0$, and $g'(0) = f(0)$.

(b) (5 points) Suppose, in addition, that f' exists and is continuous on $(-1, 1)$. Define $\varphi : (-1, 1) \to \mathbb{R}$ by

$$\varphi(x) = x^2 f(x).$$

Prove that $\varphi''(0)$ exists, and $\varphi''(0) = 2f(0)$.

8.) (10 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both Riemann integrable on $[a, b]$. Prove that $f + g$ is Riemann integrable on $[a, b]$.

You can assume the fact that a function $h : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if: for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(h, P) - L(h, P) < \epsilon$. You can also assume the fact that if a partition Q of $[a, b]$ is a refinement of a partition P of $[a, b]$ (that is, $P \subseteq Q$), then $L(h, P) \leq L(h, Q)$ and $U(h, Q) \leq U(h, P)$.