LINEAR ALGEBRA DIAGNOSTIC EXAM

AUGUST 2024

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex.

1. Let T be a linear operator on a vector space V of dimension n such that $\text{rank}(T^2) = \text{rank}(T)$. Prove that $\ker(T) \cap \text{im}(T) = \{\vec{0}\}$.

2. Suppose that $\{u_1, u_2, u_3\}$ is an orthonormal basis for an inner product space V. Prove that if v is orthogonal to both u_1 and u_2, then v is a multiple of u_3.

3. Suppose that A is a 4×6 matrix of rank 3. Prove that A has a 3×3 submatrix which is invertible. (Here, a 3×3 submatrix means the matrix left after crossing out one row and three columns.)

4. Prove that if A_1 and A_2 are two nilpotent 3×3 matrices with the same minimal polynomial, then they must be similar.

5. Let W be a subspace of an inner product space V. Then, $V = W \oplus W^\perp$ and if $v = w_1 + w_2 \in V$, with $w_1 \in W$ and $w_2 \in W^\perp$, define $T(v) = w_2 - w_1$. Prove that T is both Hermitian and unitary.

6. Let A and U be an $n \times n$ matrices, with U unitary. Suppose that $|\lambda| \leq 9$ for every eigenvalue λ of A^*A. Prove that $\|UAv\| \leq 3$ for every $n \times 1$ vector v such that $\|v\| = 1$.