Analysis Diagnostic Exam August 11, 2023

NAME: ______________________________

#1.)	____/10	#2.)	____/10	#3.)	____/10	#4.)	____/10	#5.)	____/10
#6.)	____/10	#7.)	____/10	#8.)	____/10			Total:	______/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (10 points) Let A and B be sets. Prove that

$$A \cap B = (A \cup B) \setminus [(A \setminus B) \cup (B \setminus A)].$$

Here, in general, $C \setminus D = \{x \in C : x \notin D\}$.

2.) (10 points) Prove directly from the $\epsilon - N$ definition of the limit of a sequence (that is, without using any limit theorems) that

$$\lim_{n\to\infty} \frac{4n^2 + 4n + 6}{n^2 + 1,000} = 4.$$

3.) Let $A \subseteq (0, \infty)$ be a nonempty bounded set, and let $B = \left\{ \frac{1}{a} : a \in A \right\}$.

(a) (7 points) Prove that B is bounded below and $\inf B = \dfrac{1}{\sup A}$.

(b) (3 points) Suppose in addition that A is bounded away from 0, which means that there exists $C > 0$ such that $a \geq C$ for all $a \in A$. Prove that B is bounded above, and $\sup B = \dfrac{1}{\inf A}$.

4.) (10 points) Suppose $A \subseteq \mathbb{R}$ and $y \in \mathbb{R}$. Define

$$A + y = \{a + y : a \in A\}.$$

If A is closed, prove that $A + y$ is closed.

5.) (10 points) Suppose A and B are subsets of $\mathbb{R}$. Let

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

If A and B are sequentially compact, show that $A + B$ is sequentially compact.

6.) (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let O be an open subset of $\mathbb{R}$. Prove that $f^{-1}(O)$ is open. Here continuity is defined via the $\epsilon - \delta$ definition.

7.) (10 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \sin(1/x), & \text{if } x \in \mathbb{Q}, x \neq 0 \\ x \sin(1/x), & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ denotes the rational numbers. Determine whether f is differentiable at 0 and prove your answer.

8.) (10 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is bounded below, away from 0; that is, there exists $A > 0$ such that $f(x) \geq A$, for all $x \in [a, b]$. Suppose $f \in \mathcal{R}([a, b])$. Prove that $\dfrac{1}{f} \in \mathcal{R}([a, b])$. You can assume the results of Problem 3.