Analysis Diagnostic Exam May 22, 2024

NAME: ____________________

#1.) ____/10 #2.) ____/10 #3.) ____/10 #4.) ____/10 #5.) ____/10 #6.) ____/10

#7.) ____/10 #8.) ____/10 Total: ____/80

Instructions: There are 80 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (a) (6 points) Suppose X and Y are sets, and $f : X \to Y$ is a function. If f is $1-1$, prove that $f^{-1}(f(A)) = A$ for all subsets A of X.

(b) (4 points) Give an example of sets X, Y, a function $f : X \to Y$, and a set $A \subseteq X$, such that $f^{-1}(f(A)) \neq A$. Make sure to define X, Y, A, and f, and determine $f(A)$ and $f^{-1}(f(A))$.

2.) (10 points) Suppose $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, $A \neq \emptyset$, $B \neq \emptyset$, and both A and B are bounded above. Prove that $A \cup B$ is bounded above and

$$\sup(A \cup B) = \max(\sup A, \sup B).$$

(The maximum of two numbers is just the largest of the two.)

3.) (10 points) Suppose (x_n) is a sequence of real numbers that is increasing (i.e., $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$) and bounded above. Prove that (x_n) converges to $\sup\{x_n : n \in \mathbb{N}\}$. (Do not quote the monotone sequence theorem; you are asked to prove it.)

4.) (10 points) Using only the $\epsilon - \delta$ definition of the limit (not using any theorems about the limit or continuity), show that

$$\lim_{x \to 3} \frac{2x^2 + 12}{x^2 + 21} = 1.$$

5.) Recall that for a set $C \subseteq \mathbb{R}$, the set C°, called the interior of C, consists of all points $x \in C$ such that there exists some $r > 0$ (with r depending on x) such that $(x - r, x + r) \subseteq C$.

(a) (6 points) Suppose $A, B \subseteq \mathbb{R}$. Prove that $A^\circ \cap B^\circ = (A \cap B)^\circ$.

(b) (4 points) Give an example of a countable sequence $(A_n)_{n=1}^\infty$ of subsets of $\mathbb{R}$ such that

$$\cap_{n=1}^\infty A_n^\circ \neq \left(\cap_{n=1}^\infty A_n\right)^\circ .$$

6.) (10 points) Let $E \subseteq \mathbb{R}$ with $E \neq \emptyset$. Suppose $f : \overline{E} \to \mathbb{R}$ is continuous. If f is uniformly continuous on E, prove that f is uniformly continuous on $\overline{E}$.

7.) (10 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $x_0 \in (a, b)$. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at x_0. Prove that f is continuous at x_0.

8.) Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3, & \text{if } x = 0; \\ 0, & \text{if } 0 < x < \frac{1}{2}; \\ 5, & \text{if } x = \frac{1}{2}; \\ 0, & \text{if } \frac{1}{2} < x \leq 1 \ . \end{cases}$$

For $n \in \mathbb{N}$, let $P_{2n} = \{0, \frac{1}{2n}, \frac{2}{2n}, \cdots, \frac{2n-1}{2n}, 1\}$ be the regular partition of $[0,1]$ with interval length $\Delta x_j = \frac{1}{2n}$ for all $1 \leq j \leq 2n$.

(a) (7 points) Compute $U(f, P_{2n})$ and $L(f, P_{2n})$, the upper and lower Riemann sums for f with respect to P_{2n}.

(b) (3 points) Determine whether f is Riemann integrable on $[0,1]$. Prove your answer.