LINEAR ALGEBRA DIAGNOSTIC, AUGUST 2023

All vector spaces are assumed to be finite-dimensional over the complex numbers, and all matrices are assumed to be complex, unless otherwise stated.

(1) Let V be a 4-dimensional vector space. Prove that there exist subspaces $U_1, U_2, U_3 \subset V$ such that $U_1 + U_2 + U_3 = V$, but we have strict inclusions: $U_1 + U_2 \subsetneq V$, $U_1 + U_3 \subsetneq V$, and $U_2 + U_3 \subsetneq V$.

(2) Suppose that $T\colon V \to V$ is an invertible linear transformation. If v, w, u is a linearly independent list of vectors in V, then what is the dimension of the span of $T(v), T(v+w), T(v+w+u)$?

(3) Let A be a 3×3 matrix whose eigenvalues are 0, 1, and -1 (and these are all the eigenvalues). Show that $A^5 = A$.

(4) Let T be a linear operator on a 5-dimensional vector space V. The only eigenvalue of T is -2 and $(T+2)^2$ is a non-zero operator. What are the possible Jordan canonical forms of T, up to reordering the blocks?

(5) Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation. Let v be a vector in $\mathbb{R}^5$. Show that there exists a vector w, where w is *NOT* in the range of T, such that w is orthogonal to v.

(6) Let A be a symmetric $n \times n$ real matrix and suppose that the eigenvalues of A are 1, 3, and 10 (and these are all the eigenvalues). Suppose that $v, w \in \mathbb{R}^n$ are vectors such that $\|Av\| = \|v\|$ and $\|Aw\| = 10\|w\|$. Show that v is orthogonal to w.