# LINEAR ALGEBRA DIAGNOSTIC EXAM

MAY 2024

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex.

**1.** Let $V$, $W$, and $U$ be $n$-dimensional vector spaces and let $T : V \to W$ and $S : W \to U$ be linear transformations such that $ST = 0$. Prove that $\operatorname{rank}(T) + \operatorname{rank}(S) \leq n$.

**2.** Let $V$ be a 4-dimensional vector space containing 3-dimensional subspaces $U$ and $W$. Prove that there exists a basis $\{v_1, v_2, v_3, v_4\}$ for $V$ such that $v_1, v_2 \in U \cap W$.

**3.** Let $\{v_1, v_2, v_3\}$ be an linearly independent set of three vectors in the inner product vector space $V$. Prove that there is a vector $w \in V$ such that $\langle w, v_1 \rangle = \langle w, v_2 \rangle = 0$ and $\langle w, v_3 \rangle = 1$.

**4.** Let $A$ be an $n \times n$ matrix and suppose that $v$ is a non-zero eigenvector for both $A$ and $A^*$. Prove that the eigenvalue corresponding to $v$ for $A$ is the complex conjugate of the eigenvalue corresponding to $v$ for $A^*$.

**5.** Let $V$ be a 4-dimensional vector space and let $T : V \to V$ be an operator with minimal polynomial equal to $x^4$. Prove that there is a vector $v \in V$ such that $\{v, T(v), T^2(v), T^3(v)\}$ is a basis of $V$.
[**Hint:** There is a basis $\mathcal{B}$ such that the matrix $[T]_{\mathcal{B}}$ is in Jordan form.]

**6.** Suppose that $A$ is a $5 \times 5$ matrix and $-1$, $2$, and $3$ are eigenvalues of $A$ (but there may be others). If $A$ has rank 4 and is not diagonalizable, then what are the possible Jordan canonical forms of $A$ (up to reordering the blocks)?